New Foundations of the Theory of Elliptic Functions *

Carl Gustav Jacob Jacobi

Prooemium

Almost two years ago, when it was customary to examine the theory of elliptic functions in greater detail, I stumbled upon certain most important questions which seemed both to create a new branch of this theory and promote the art of analysis significantly. Having given a satisfactory and because of the inherent difficulty hardly expected answer to those questions, I communicated the first major results, at first in short form and without a proof, then, because soon afterwards the proof seemed to be desired even more and, after new discoveries, those results seemed to be seen suspiciously, with a proof with the geometers. At the same time, I was urged to publish the complete list of question I studied. To satisfy this desire at least partly I decided to publish the foundations on which my investigations are based. Now, we commend these new foundations of the theory of elliptic functions to the indulgence of the geometers.

Written February, 1829 at the University of Königsberg

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*Original title: "' Fundamenta Nova Theoriae Functionum Ellipticarum"', first published in "'*Königsberg: Gebrüder Borntraeger*, 1829, as a book, reprinted in "'*C.G.J. Jacobi's Gesammelte Werke Volume 1*, pp. 49-239 , translated by: Alexander Aycock, for the "'Euler-Kreis Mainz"'

1 On the Transformation of Elliptic Functions

1.1 Exposition of the general Problem on the Transformation

1.

The most memorable integrals which are exhibited by the formula $\int \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$ and which constitute the first kind of elliptic functions, as they are called nowadays, depend on two arguments, on the amplitude φ and the modulus k. Having compared the values of functions of this kind they take on while the modulus stays fixed, the Analysists had discovered many extraordinary results concerning their addition and multiplication. We, with great admiration, have recently seen that this theory was promoted significantly by Abel in his treatise (Crelle Journal für reine und angewandte Mathematik Vol. II).

Another question of not minor importance - understood in the broadest sense it even contains the first - is the question on the comparison of the elliptic functions for different moduli. After the beautiful discoveries of Legendre - the founder of the theory of elliptic functions - we, at first, reduced this question to certain principles and gave their general solution (Astronomische Nachrichten, 1827, n°123, 127). And we now want to explain this theory on the transformation and all the results following from this for the analysis of elliptic functions in great detail.

2.

The general problem we want to tackle is the following:

"To find a rational function y of the variable x of such a kind that we have

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}."$$

We see that this problem contains both the multiplication and the transformation.

Examples of rational functions y of such a kind solving the given problem have been known for a long time. At first, it was known, no matter which odd number n was given, that one can exhibit a rational function y of such a kind that we have:

$$\frac{dy}{\sqrt{A + By + Cy^2 + Dy^3 + Ey^4}} = \frac{ndx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}};$$

this is the theorem on multiplication. For this aim, one has to assume this form:

$$y = \frac{a + a'x + a''x^2 + a'''x^3 + \cdots + a^{(nn)}x^{nn}}{b + b'x + b''x^2 + b'''x^3 + \cdots + b^{(nn)}x^{nn}},$$

after having determined the coefficients $a, a', a'', \cdots, ; b, b', b'', \cdots$ in the right manner. Additionally, it has also been known for a long time that this form

$$y = \frac{a + a'x + a''x^2}{b + b'x + b''x^2},$$

or this more general:

$$y = \frac{a + a'x + a''x^2 + a'''x^3 + \cdots + a^{(2^m)}x^{2^m}}{b + b'x + b''x^2 + b'''x^3 + \cdots + b^{(2^m)}x^{2^m}},$$

which results from the preceding by iterated substitution, can be determined in such a way that it solves the problem. Recently, it was even proved by

Legendre that for this aim this form, determined in the correct way, of course, can be used:

$$y = \frac{a + a'x + a''x^2 + a'''x^3 + \cdots + a^{(3^m)}x^{3^m}}{b + b'x + b''x^2 + b'''x^3 + \cdots + b^{(3^m)}x^{3^m}}.$$

Combining these two forms it is plain that the problem can be solved by an appropriate choice for the coefficients and by putting:

$$y = \frac{a + a'x + a''x^2 + a'''x^3 + \cdots + a^{(p)}x^p}{b + b'x + b''x^2 + b'''x^3 + \cdots + b^{(p)}x^p},$$

if p is a number of the form $2^\alpha 3^\beta (2m+1)^2$. Now, it will be proved in the following that the same holds, *no matter what number p is.*

1.2 The Principles of the Transformation

3.

Let us denote two polynomial functions of the variable x by U, V, further, let $y = \frac{U}{V}$, it is:

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{VdU - UdV}{\sqrt{Y}},$$

having for the sake of brevity put:

$$Y = A'V^4 + B'V^3U + C'V^2U^2 + D'VU^3 + E'U^4.$$

The fraction $\frac{VdU-UdV}{\sqrt{Y}}$ can be transformed into a simpler form, if Y contains multiple factors; when except for four mutually different linear factors two of the remaining ones are equal, the fraction, by itself, reduces to the differential of an elliptic function $\frac{dx}{M\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}}$, where M denotes a function of the variable x. Let us examine this case in more detail and see, how many and which conditions it demands.

Let U, V be functions, the one of p-th order, the other of m-th order such that $m \leq p$; Y will be of degree $4p$. Now, that, having excluded the case of four linear factors, of the remaining factors of the function Y, whose number is $4p - 4$, two become equal to each other, $2p - 2$ conditions are to be satisfied. For, so many conditional equations between its coefficients must hold as many double linear factors the given function must have.

But the functions U,V contain $m+p+2$ or rather $m+p+1$, because one of them can be set $=1$, undetermined constant quantities. Therefore, their total amount either becomes equal to the number of $2p-2$ conditions or the number of conditions is smaller than the number of undetermined quantities; let us suppose that m is any of the numbers $p-3$, $p-2$, $p-1$, p in which cases the number of unknowns becomes $2p-2$, $2p-1$, $2p$, $2p+1$, respectively. It will be shown below that the first two cases are to be neglected and this is already plain by the following argument. For, having found the functions U,V providing the function Y with the prescribed form, if one puts $\alpha+\beta x$ instead of x, neither the structure of the functions U, V, Y nor the number of double factors of the function Y is changed: Hence it is possible to introduce two arbitrary quantities from the beginning. Therefore, the number of undetermined quantities has to exceed the number of conditions by at least two, whence the cases $m=p-3$ and $m=p-2$ are to be neglected. Further, having put $\frac{\alpha+\beta x}{1+\gamma x}$ for x we see that the third case can be reduced the fourth and the fourth is not changed by any means, in which case therefore three of the unknowns remain arbitrary and have to stay arbitrary.

Now, it is therefore shown, what can be concluded from the comparison of the number of undetermined quantities to the number of conditions: *No matter what the number p is, the form:*

$$y=\frac{a+a'x+a''x^2+a'''x^3+\cdots+a^{(p)}x^p}{b+b'x+b''x^2+b'''x^3+\cdots+b^{(p)}x^p}$$

can be determined in such a way that we have:

$$\frac{dy}{\sqrt{A'+B'y+C'y^2+D'y^3+E'y^4}}=\frac{dx}{M\sqrt{A+Bx+Cx^2+Dx^3+Ex^4}},$$

where M denotes a rational function of x: the solution can involve up to three arbitrary constants.

4.

In order to determine the function M, let

$$Y=(A+Bx+Cx^2+Dx^3+Ex^4)TT,$$

where T denotes a polynomial function of the variable x: It will be

$$M = \frac{T}{V\frac{dU}{dx} - U\frac{dV}{dx}}.$$

T will be of order $2p-2$ and $V\frac{dU}{dx} - U\frac{dV}{dx}$ cannot be of higher order. Now, it is known in certain cases, of course, whenever the number p has the form $2^{\alpha}3^{\beta}(2n+1)^2$, that M becomes even constant. The same will be proved in the following, for every number p.

We can assume that the functions U, V do not have a common factor; for, having assumed a common factor, the fraction $\frac{U}{V} = y$ is not changed. Let us resolve the expression

$$A' + B'y + C'y^2 + D'y^3 + E'y^4$$

into linear factors such that we have:

$$A' + B'y + C'y^2 + D'y^3 + E'y^4 = A'(1-\alpha'y)(1-\beta'y)(1-\gamma'y)(1-\delta'y),$$

whence it is:

$$Y = A'V^4 + B'V^3U + C'V^2U^2 + D'VU^3 + E'U^4 = A'(V-\alpha'U)(V-\beta'U)(V-\gamma'U)(V-\delta'U).$$

Now, there cannot exist a factor which is a common factor of all the quantities $V-\alpha'U$, $V-\beta'U$, $V-\gamma'U$, $V-\delta'U$ or even only two of them; for, this factor would divide V and U at the same time, which we assumed to have no common factor. Therefore, if any linear factor divides the function Y twice, the same has to divide one of the quantities $V-\alpha'U$, $V-\beta'U$, $V-\gamma'U$, $V-\delta'U$ twice.

Now, let us consider the following equations:

$$\begin{aligned}
(V-\alpha'U)\frac{dU}{dx} - \frac{V-\alpha'U}{dx}U &= V\frac{dU}{dx} - U\frac{dV}{dx} \\
(V-\beta'U)\frac{dU}{dx} - \frac{V-\beta'U}{dx}U &= V\frac{dU}{dx} - U\frac{dV}{dx} \\
(V-\gamma'U)\frac{dU}{dx} - \frac{V-\gamma'U}{dx}U &= V\frac{dU}{dx} - U\frac{dV}{dx} \\
(V-\delta'U)\frac{dU}{dx} - \frac{V-\delta'U}{dx}U &= V\frac{dU}{dx} - U\frac{dV}{dx},
\end{aligned}$$

from which it follows that a factor dividing any of the quantities $V - \alpha' U$, $V - \beta' U$, $V - \gamma' U$, $V - \delta' U$ and hence also its differential twice also divides the expression $V\frac{dU}{dx} - U\frac{dV}{dx}$. But, we put the product conflated of all these factors, also dividing Y twice, $= T$, whence T will divide $V\frac{dU}{dx} - U\frac{dV}{dx}$. But T is not of lower order than $V\frac{dU}{dx} - U\frac{dV}{dx}$, whence we see that

$$M = \frac{T}{V\frac{dU}{dx} - U\frac{dV}{dx}}$$

becomes a constant.

Additionally, we want to mention, if the one of the functions U, V would have been of lower order than $p - 1$, that then also $V\frac{dU}{dx} - U\frac{dV}{dx}$ would have been of lower order than T, which nevertheless has to divide the latter; since this is absurd, the cases $m = p - 2$, $m = p - 3$ must be neglected.

Therefore, it is now proved *that the form*

$$y = \frac{a + a'x + a''x^2 + \cdots + a^{(p)}x^p}{b + b'x + b''x^2 + \cdots + b^{(p)}x^p},$$

for any number p, can be determined in such a way that the following identity results:

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}} = \frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}},$$

This is the fundamental principle in the theory of transformations of elliptic functions.

1.3 It is propounded to reduce the Expression $\frac{dy}{\sqrt{\pm(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$ to the simpler Form $\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$.

5.

By means of three arbitrary constants we saw our solution to admit that the expression $A + Bx + Cx^2 + Dx^3 + Ex^4$ can be transformed into this simpler one: $A(1 - x^2)(1 - k^2x^2)$. To illustrate this and the remaining things which were demonstrated by an example let the expression given in the title be propounded:

$$\frac{dy}{\sqrt{\pm(y - \alpha)(y - \beta)(y - \gamma)(y - \delta)}}$$

which after the substitution

$$y = \frac{a + a'x + a''x^2}{b + b'x + b''x^2},$$

is to be transformed into this simpler one

$$\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$

The question is about the determination of the substitution to be made and about the modulus k and the constant factor M from the given quantities α, β, γ, δ.

Let us put $a + a'x + a''x^2 = U$, $b + b'x + b''x^2 = V$, $y = \frac{U}{V}$; from the principles just explained it has to be:

$$(U-\alpha V)(U-\beta V)(U-\gamma V)(U-\delta V) = K(1-x^2)(1-k^2x^2)(1+mx)^2(1+nx)^2,$$

where K denotes an arbitrary constant. Therefore, we see that two of the total amount of factors $V - \alpha' U$, $V - \beta' U$, $V - \gamma' U$, $V - \delta' U$, which will be of second order, even become squares. Therefore, let us put:

$$\begin{aligned} U - \gamma V &= C(1+mx)^2 \\ U - \delta V &= D(1+nx)^2. \end{aligned}$$

Concerning the remaining functions $U - \alpha V, U - \beta V$, one can either put:

$$U - \alpha V = A(1-x^2), \qquad U - \beta V = B(1-k^2x^2)$$

or:

$$U - \alpha V = A(1-x)(1-kx), \quad U - \beta V = B(1+x)(1+kx),$$

where A, B, C, D denote constant quantities. The first possibility would have to be neglected. For, it will yield $\frac{U-\alpha V}{U-\beta V} = \frac{y-\alpha}{y-\beta} = \frac{A}{B} \cdot \frac{1-x^2}{1-k^2x^2}$, whence having transformed x into $-x$ it would follow that y remains unchanged; that this is absurd is obvious from the equations:

$$\frac{U-\alpha V}{U-\gamma V} = \frac{y-\alpha}{y-\gamma} = \frac{A}{C}\cdot\frac{1-x^2}{(1+mx)^2}$$
$$\frac{U-\alpha V}{U-\delta V} = \frac{y-\alpha}{y-\delta} = \frac{A}{D}\cdot\frac{1-x^2}{(1+nx)^2}.$$

Therefore, one has to put:

$$\begin{aligned}
&(1) \qquad U-\alpha V = A(1-x)(1-kx)\\
&(2) \qquad U-\beta V = A(1+x)(1+kx)\\
&(3) \qquad U-\gamma V = C(1+mx)^2\\
&(4) \qquad U-\delta V = D(1+nx)^2.
\end{aligned}$$

It should be mentioned that one of the constants A, B, C, D can be determined arbitrarily.

6.

From equation (1), having put $x = 1$ and $x = \frac{1}{k}$, we see that $U = \alpha V$. Hence from the equation:

$$\frac{U-\gamma V}{U-\beta V} = \frac{C}{B}\cdot\frac{(1+mx)^2}{(1+x)(1+kx)},$$

after having put $x = 1$, it results:

$$\frac{\alpha-\gamma}{\alpha-\beta} = \frac{C}{B}\cdot\frac{(1+m)^2}{2(1+k)},$$

and for $x = \frac{1}{k}$:

$$\frac{\alpha-\gamma}{\alpha-\beta} = \frac{C}{B}\cdot\frac{(1+\frac{m}{k})^2}{2(1+\frac{1}{k})},$$

whence it is

$$(1+m)^2 = k\left(1+\frac{m}{k}\right)^2.$$

Further, one will find in like manner:

$$(1+n)^2 = k\left(1+\frac{n}{k}\right)^2,$$

whence $m = \sqrt{k}$, $n = -\sqrt{k}$. Therefore, it is not possible to assume m and n to be equal; since then the expression $\frac{U-\gamma V}{U-\delta V} = \frac{y-\gamma}{y-\delta}$ and hence y would be a constant.

Now, in the equation

$$\frac{U-\gamma V}{U-\delta V} = \frac{y-\gamma}{y-\delta} = \frac{C}{D}\cdot\left\{\frac{1+\sqrt{k}\cdot x}{1-\sqrt{k}\cdot x}\right\}^2$$

let us at first put $x = 1$ in which case $U = \alpha V$, and then $x = -1$ in which case $U = \beta V$. The following two equations result:

$$\frac{\alpha-\gamma}{\alpha-\delta} = \frac{C}{D}\cdot\left\{\frac{1+\sqrt{k}}{1-\sqrt{k}}\right\}^2$$
$$\frac{\beta-\gamma}{\beta-\delta} = \frac{C}{D}\cdot\left\{\frac{1-\sqrt{k}}{1+\sqrt{k}}\right\}^2.$$

After having multiplied those equations by each other we have:

$$\frac{C}{D} = \sqrt{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}},$$

whence it is possible to put:

$$C = \sqrt{(\alpha-\gamma)(\beta-\gamma)}$$
$$D = \sqrt{(\alpha-\delta)(\beta-\delta)};$$

for, one of the quantities A, B, C, D could be determined arbitrarily.

From the same equations, having divided one by the other, we will obtain:

$$\frac{1+\sqrt{k}}{1-\sqrt{k}} = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}}{\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}},$$

whence it follows:

$$\sqrt{k} = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} - \sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} + \sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}.$$

Finally, let us note the formula:

$$\sqrt{k} + \frac{1}{\sqrt{k}} = 2 \cdot \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} + \sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} - \sqrt[4]{(\alpha-\delta)(\beta-\gamma)}},$$

whence we obtain:

$$\begin{aligned}
(1-\sqrt{k})\left(1-\frac{1}{\sqrt{k}}\right) &= \frac{-4\sqrt{(\alpha-\delta)(\beta-\delta)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)} - \sqrt{(\alpha-\delta)(\beta-\gamma)}} \\
(1+\sqrt{k})\left(1+\frac{1}{\sqrt{k}}\right) &= \frac{4\sqrt{(\alpha-\delta)(\beta-\delta)}}{\sqrt{(\alpha-\gamma)(\beta-\delta)} - \sqrt{(\alpha-\delta)(\beta-\gamma)}}.
\end{aligned}$$

In order to determine the constants A, B observe that from the equations (1), (2), (3), after having put $x = \frac{1}{\sqrt{k}}$ which in turn leads to $U = \delta V$, it is found:

$$\begin{aligned}
\frac{\delta-\alpha}{\delta-\gamma} &= \frac{A(1-\sqrt{k})\left(1-\sqrt{\frac{1}{k}}\right)}{4\sqrt{(\alpha-\gamma)(\beta-\gamma)}} \\
\frac{\delta-\beta}{\delta-\gamma} &= \frac{B(1+\sqrt{k})\left(1+\sqrt{\frac{1}{k}}\right)}{4\sqrt{(\alpha-\gamma)(\beta-\gamma)}},
\end{aligned}$$

whence it follows:

$$\begin{aligned}
A &= -\frac{\sqrt{(\alpha-\gamma)(\alpha-\delta)}}{\gamma-\delta}\left\{\sqrt{(\alpha-\gamma)(\beta-\delta)} - \sqrt{(\alpha-\delta)(\beta-\gamma)}\right\} \\
B &= \frac{\sqrt{(\beta-\gamma)(\beta-\delta)}}{\gamma-\delta}\left\{\sqrt{(\alpha-\gamma)(\beta-\delta)} - \sqrt{(\alpha-\delta)(\beta-\gamma)}\right\}.
\end{aligned}$$

7.

From the general principle we established above it follows that in our example the expression $V\frac{dU}{dx} - U\frac{dV}{dx}$ will be equal to the product $(1+\sqrt{k}\cdot x)(1-\sqrt{k}\cdot x)$ multiplied by a constant, which claim is proved by direct calculation.

It, as it is plain from the expansion, is:

$$(\gamma-\delta)\left(U\frac{dV}{dx} - V\frac{dU}{dx}\right) = (U-\gamma V)\frac{d(U-\delta V)}{dx} - (U-\delta V)\frac{d(U-\gamma V)}{dx}.$$

But we obtained:

$$\begin{aligned} U-\gamma V &= C(1+\sqrt{k}\cdot x)^2 \\ U-\delta V &= D(1-\sqrt{k}\cdot x)^2, \end{aligned}$$

whence it follows:

$$\begin{aligned} \frac{d(U-\gamma V)}{dx} &= \quad 2C(1+\sqrt{k}\cdot x)\sqrt{k} \\ \frac{d(U-\delta V)}{dx} &= -2D(1-\sqrt{k}\cdot x)\sqrt{k}. \end{aligned}$$

Hence it results:

$$(\gamma-\delta)\left(V\frac{dU}{dx} - U\frac{dV}{dx}\right) = 4\sqrt{k}\cdot CD(1+\sqrt{k}\cdot x)(1-\sqrt{k}\cdot x).$$

Having gathered all these in the right way we obtain:

$$\frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{4\sqrt{k}}{\gamma-\delta}\cdot\sqrt{\frac{CD}{-AB}}\cdot\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

whence it follows:

$$\begin{aligned} M = \frac{\gamma-\delta}{4\sqrt{k}}\sqrt{\frac{-AB}{CD}} &= \frac{\sqrt{(\alpha-\gamma)(\beta-\delta)} - \sqrt{(\alpha-\delta)(\beta-\gamma)}}{4\sqrt{k}} \\ &= \left\{\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} - \sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}\right\}^2, \end{aligned}$$

$$\frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$= \frac{dx}{\sqrt{[1-x^2]\left[\left(\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}+\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}\right)^4 - \left(\frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)}-\sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}\right)^4 x^2\right]}}.$$

Having put $(\alpha-\gamma)(\beta-\delta) = G, (\alpha-\delta)(\beta-\gamma) = G'$ this becomes:

$$\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{dx}{\sqrt{[1-x^2]\left[\left(\frac{\sqrt[4]{G}+\sqrt[4]{G'}}{2}\right)^4 - \left(\frac{\sqrt[4]{G}-\sqrt[4]{G'}}{2}\right)^4\right]}}.$$

Let $G = mm$, $G' = nn$, further let:

$$m' = \frac{1}{2}(m \quad + n), \quad n' = \sqrt{mn}$$
$$m'' = \frac{1}{2}(m' + n'), \quad n'' = \sqrt{m'n'};$$

having put $x = \sin\varphi$ it will be:

$$\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{d\varphi}{\sqrt{m''m''\cos^2\varphi + n''n''\sin^2\varphi}}.$$

Additionally, the value of x is easily calculated by means of the formula:

$$\frac{1-\sqrt{k}\cdot x}{1+\sqrt{k}\cdot x} = \sqrt[4]{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}} \cdot \sqrt{\frac{y-\delta}{y-\gamma}},$$

where:

$$\sqrt{k} = \frac{\sqrt[4]{G}-\sqrt[4]{G'}}{\sqrt[4]{G}+\sqrt[4]{G'}} = \sqrt[4]{\frac{m''m''-n''n''}{m''m''}}.$$

8.

The quantities $\alpha, \beta, \gamma, \delta$ can be interchanged arbitrarily in the propounded formulas. This certainly is to our advantage and, whenever a condition is added, if it is possible, of course, the transformation succeeds by means of a real substitution. Let us examine this in more detail.

Let us put that the quantities $\alpha, \beta, \gamma, \delta$ are all real, further let $\alpha > \beta > \gamma > \delta$ such that $\alpha - \beta, \alpha - \gamma, \alpha - \delta$ are real positive quantities. Now, one has to distinguish the boundaries within which the value of y is contained:

$$1) \ \delta \ \text{and} \ \gamma, \quad 2) \ \gamma \ \text{and} \ \beta, \quad 3) \ \beta \ \text{and} \ \alpha, \quad 4) \ \alpha \ \text{and} \ \delta.$$

In the last case, imagine that the transition from α to δ happens through infinity. We see that the expression

$$\frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$$

becomes real only in the second and fourth case, whereas the expression

$$\frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}}$$

becomes real only in the first and the third case. Table I indicates real substitutions corresponding to the four cases. The second table II contains the formulas providing the transformation of

$$\frac{dy}{\sqrt{\pm(y-\alpha)(y-\beta)(y-\gamma)}}$$

into a simpler form by means of a substitution for the boundaries within which the value of the argument y is contained:

$$1) \ -\infty \ \text{and} \ \gamma, \quad 2) \ \gamma \ \text{and} \ \beta, \quad 3) \ \beta \ \text{and} \ \alpha, \quad 4) \ \alpha \ \text{and} \ +\infty.$$

These formulas, by diving by $-\delta$ under the square root sign and putting $\delta = \infty$, can easily be derived from table I.

Table I.

$$(A.)\qquad \frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{dx}{\sqrt{(1-x)(L^4-N^4x^2)}}$$

$$L = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} + \sqrt[4]{(\alpha-\beta)(\gamma-\delta)}}{2}, \quad N = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} - \sqrt[4]{(\alpha-\beta)(\gamma-\delta)}}{2}$$

(I.) Limits: $\alpha \cdots \pm\infty \cdots \delta : \dfrac{L-Nx}{L+Nx} = \sqrt[4]{\dfrac{(\alpha-\beta)(\beta-\delta)}{(\alpha-\gamma)(\gamma-\delta)}} \cdot \sqrt{\dfrac{y-\gamma}{y-\beta}}$

(II.) Limits: $\alpha \cdots\cdots\cdots\cdots \delta : \dfrac{L-Nx}{L+Nx} = \sqrt[4]{\dfrac{(\beta-\delta)(\gamma-\delta)}{(\alpha-\beta)(\alpha-\gamma)}} \cdot \sqrt{\dfrac{\alpha-y}{y-\delta}}.$

$$(B.)\qquad \frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{dx}{\sqrt{(1-x)(L^4-N^4x^2)}}$$

$$L = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} + \sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}, \quad N = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\delta)} - \sqrt[4]{(\alpha-\delta)(\beta-\gamma)}}{2}$$

(I.) Limits: $\beta \cdots\cdots \alpha : \dfrac{L-Nx}{L+Nx} = \sqrt[4]{\dfrac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}} \cdot \sqrt{\dfrac{y-\delta}{y-\gamma}}$

(II.) Limits: $\delta \cdots\cdots \gamma : \dfrac{L-Nx}{L+Nx} = \sqrt[4]{\dfrac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}} \cdot \sqrt{\dfrac{\beta-y}{\alpha-y}}.$

Table II.

$$(A.)\qquad \frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)}} = \frac{dx}{\sqrt{(1-x)(L^4-N^4x^2)}}$$

$$L = \frac{\sqrt[4]{\alpha-\gamma} + \sqrt[4]{\alpha-\beta}}{2}, \quad N = \frac{\sqrt[4]{\alpha-\gamma} - \sqrt[4]{(\alpha-\beta}}{2}$$

(I.) Limits: $\alpha \cdots \pm\infty : \dfrac{L-Nx}{L+Nx} = \sqrt[4]{\dfrac{\alpha-\beta}{\alpha-\gamma}} \cdot \sqrt{\dfrac{y-\gamma}{y-\beta}}$

(II.) Limits: $\gamma \cdots \pm\beta : \dfrac{L-Nx}{L+Nx} = \dfrac{\sqrt{\alpha-y}}{\sqrt[4]{(\alpha-\beta)(\alpha-\gamma)}}.$

$$(B.)\qquad \frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)}} = \frac{dx}{\sqrt{(1-x)(L^4-N^4x^2)}}$$

$$L = \frac{\sqrt[4]{\alpha-\gamma}+\sqrt[4]{\beta-\gamma}}{2}, \quad N = \frac{\sqrt[4]{\alpha-\gamma}-\sqrt[4]{(\beta-\gamma}}{2}$$

$$\text{(I.)}\qquad \text{Limits:}\quad \beta\cdots\cdots\alpha : \frac{L-Nx}{L+Nx} = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\gamma)}}{\sqrt{y-\gamma}}$$

$$\text{(II.)}\qquad \text{Limits:}\quad -\infty\cdots\gamma : \frac{L-Nx}{L+Nx} = \sqrt[4]{\frac{\alpha-\gamma}{\beta-\gamma}}\cdot\sqrt{\frac{\beta-y}{\alpha-y}}.$$

In these formulas for the given boundaries as x goes over from -1 to 1 at the same time y goes over from the one boundary to the other. But having commuted the boundaries corresponding to the formulas (I.) and (II.) we see that the expression $\frac{L-Nx}{L+Nx}$ creates an imaginary value of the form $\pm iR$, where we put $i=\sqrt{-1}$ and assume R to denote a real quantity; additionally, we see that x takes on the form $\frac{Le^{i\varphi}}{N} = \frac{e^{i\varphi}}{\sqrt{k}}$, whence it follows

$$\frac{L-Nx}{L+Nx} = \frac{1-e^{i\varphi}}{1+e^{i\varphi}} = \frac{e^{-\frac{i\varphi}{2}}-e^{\frac{i\varphi}{2}}}{e^{-\frac{i\varphi}{2}}+e^{\frac{i\varphi}{2}}} = -i\tan\frac{\varphi}{2}.$$

Let us substitute the form we were led to in this occasion, $x=\frac{e^{i\varphi}}{\sqrt{k}}$, in the expression $\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$. Hence it results:

$$\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{ie^{i\varphi}d\varphi}{\sqrt{k}\cdot\sqrt{\left(1-\frac{e^{2i\varphi}}{k}\right)\left(1-e^{2i\varphi}\right)}} = \frac{d\varphi}{\sqrt{(1-ke^{2i\varphi})(1-ke^{-2i\varphi})}}$$

$$= \frac{d\varphi}{\sqrt{1-2k\cos 2\varphi+kk}} = \frac{d\varphi}{\sqrt{(1-k)^2\cos^2\varphi+(1+k)^2\sin^2\varphi}}.$$

This substitution is certainly remarkable. For, by putting $x=\sin\psi$ from it this even more general formula follows:

$$\frac{k^n\sin^{2n}\psi d\psi}{\sqrt{1-k^2\sin^2\psi}} = \frac{(\cos 2n\varphi+i\sin 2n\varphi)d\varphi}{\sqrt{1+2k\cos 2\varphi+kk}},$$

whence, since the imaginary part vanishes, for the boundaries 0 and π one obtains:

$$\int_0^{\pi} \frac{k^n \sin^{2n} \psi d\psi}{\sqrt{1-k^2 \sin^2 \psi}} = \int_0^{\pi} \frac{\cos 2n\varphi d\varphi}{\sqrt{1-2k\cos 2\varphi + kk}} = \int_0^{\pi} \frac{\cos n\varphi d\varphi}{\sqrt{1-2k\cos \varphi + kk}},$$

which is a short proof for the remarkable formula given by Legendre. From the tables I. and II. it it possible to derive two others after having commuted the boundaries within which the valor of y is contained and having put $x = \frac{Le^{i\varphi}}{N}$. For the assigned boundaries the angle φ grows from 0 to π, whereas y goes over from the one boundary to the other.

Table III.

$(A.)$ $$\frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}}$$

$$m = \sqrt[4]{(\alpha-\gamma)(\beta-\delta)(\alpha-\beta)(\gamma-\delta)}, \quad n = \frac{\sqrt{(\alpha-\gamma)(\beta-\gamma)} + \sqrt{(\alpha-\beta)(\gamma-\delta)}}{2}$$

(I.) Limits: $\gamma \cdots\cdots\cdots\cdots \beta$: $\tan\frac{\varphi}{2} = \sqrt[4]{\frac{(\alpha-\beta)(\beta-\delta)}{(\alpha-\gamma)(\gamma-\delta)}} \cdot \sqrt{\frac{y-\gamma}{\beta-y}}$

(II.) Limits: $\alpha \cdots \pm\infty \cdots \delta$: $\tan\frac{\varphi}{2} = \sqrt[4]{\frac{(\beta-\delta)(\gamma-\delta)}{(\alpha-\beta)(\alpha-\gamma)}} \cdot \sqrt{\frac{y-\gamma}{y-\delta}}.$

$(B.)$ $$\frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)(y-\delta)}} = \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}}$$

$$m = \sqrt[4]{(\alpha-\gamma)(\beta-\delta)(\alpha-\delta)(\beta-\gamma)}, \quad n = \frac{\sqrt{(\alpha-\gamma)(\beta-\delta)} + \sqrt{(\alpha-\delta)(\beta-\gamma)}}{2}$$

(I.) Limits: $\delta \cdots\cdots\cdots\cdots \gamma$: $\tan\frac{\varphi}{2} = \sqrt[4]{\frac{(\alpha-\gamma)(\beta-\gamma)}{(\alpha-\delta)(\beta-\delta)}} \cdot \sqrt{\frac{y-\delta}{\gamma-y}}$

(II.) Limits: $\beta \cdots\cdots\cdots\cdots \alpha$: $\tan\frac{\varphi}{2} = \sqrt[4]{\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}} \cdot \sqrt{\frac{y-\beta}{\alpha-y}}.$

Table IV.

$$(A.)\qquad \frac{dy}{\sqrt{(y-\alpha)(y-\beta)(y-\gamma)}} = \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}}$$

$$m = \sqrt[4]{(\alpha-\gamma)(\alpha-\beta)}, \quad n = \frac{\sqrt{\alpha-\gamma}+\sqrt{\alpha-\beta}}{2}$$

$$\text{(I.)}\qquad \text{Limits:}\quad \gamma\cdots\cdots\beta: \quad \tan\frac{\varphi}{2} = \sqrt[4]{\frac{\alpha-\beta}{\alpha-\gamma}}\cdot\sqrt{\frac{y-\gamma}{\beta-y}}$$

$$\text{(II.)}\qquad \text{Limits:}\quad \alpha\cdots+\infty: \quad \tan\frac{\varphi}{2} = \frac{\sqrt{y-\alpha}}{\sqrt[4]{(\alpha-\beta)(\alpha-\gamma)}}.$$

$$(B.)\qquad \frac{dy}{\sqrt{-(y-\alpha)(y-\beta)(y-\gamma)}} = \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}}$$

$$m = \sqrt[4]{(\alpha-\gamma)(\beta-\gamma)}, \quad n = \frac{\sqrt{\alpha-\gamma}+\sqrt{\beta-\gamma}}{2}$$

$$\text{(I.)}\qquad \text{Limits:}\quad -\infty\cdots\gamma: \quad \tan\frac{\varphi}{2} = \frac{\sqrt[4]{(\alpha-\gamma)(\beta-\gamma)}}{\sqrt{\gamma-y}}$$

$$\text{(II.)}\qquad \text{Limits:}\quad \beta\cdots\cdots\cdots\alpha: \quad \tan\frac{\varphi}{2} = \sqrt[4]{\frac{\alpha-\gamma}{\beta-\gamma}\cdot\frac{y-\beta}{\alpha-y}}.$$

We treated this question in more detail to have a fully worked out example. The cases where either two or four of the quantities $\alpha, \beta, \gamma, \delta$ are imaginary still remain. The first case also admits a real solution which does not contain imaginary quantities at all. The second case does not allow such a solution at all. Therefore, to reduce everything to real numbers a new transformation will be necessary, whence the desired beauty of the formulas gets lost. Therefore, we will not address this question.

To the propounded substitution corresponds another formula, inverse to it, of the form

$$x = \frac{a + a'y + a''y^2}{b + b'y + b''y^2},$$

which yields most elegant formulas itself. But, that it does not seem that we might stay at this question too long, we want to postpone the investigation to another occasion. We return to general questions.

1.4 On the Transformation of the Expression $\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}}$ into another similar one $\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$

10.

We saw that the given expression

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + E'y^4}}$$

by means of a transformation of this kind:

$$y = \frac{a + a'x + a''x^2 + \cdots + a^{(p)}x^p}{b + b'x + b''x^2 + \cdots + b^{(p)}x^p} = \frac{U}{V},$$

no matter what number p is, can be transformed into another one similar to it:

$$\frac{dx}{\sqrt{A + Bx + Cx^2 + Dx^3 + Ex^4}}.$$

A substitution of such a kind depends on the coefficients A', B', C', D', E' and crucially on the number p which denotes the exponent of the highest order found in the rational functions U, V. Therefore, in the following we will say that a substitution or transformation *is of $p-$th order belongs to the $p-th$ order or, simpler, corresponds to the number p.*

Now, intending to examine the nature of this substitution in more detail, let us put that more complex form aside:

$$\frac{dy}{\sqrt{A' + B'y + C'y^2 + D'y^3 + D'y^4}},$$

and let us discuss, how to transform this simpler form $\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}}$ to which - as we saw and as it is known - the latter can be reduced, into another similar one $\frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$.

Having examined the nature of the propounded equation carefully it is found that the problem is solved if one of the functions U, V is odd and the other even; this is already indicated by the examples explored by the analysts up to now. For this task, one has distinguish the case in which the order of the odd function is smaller and the case in which the order of the even function is smaller, and the case in which the transformation belongs to an even number and the case in which the transformation is of odd order.

Now, let us therefore *at first* prove that the transformation succeeds, if a transformation of even order or of the following form is used:

$$y = \frac{x(a + a'x^2 + a''x^4 + a^{(m-1)}x^{2m-2})}{a + b'x^2 + b''x^4 + b^{(m)}x^{2m}} = \frac{U}{V}.$$

Here, the functions $V + U, V - U, V + \lambda U, V - \lambda U$ will all be of even order, whence we want to put:

$$\begin{array}{lll}
(1.) & V + U & = (1+x)(1+kx)AA \\
(2.) & V - U & = (1-x)(1-kx)BB \\
(3.) & V + \lambda U & = CC \\
(4.) & V + \lambda U & = DD,
\end{array}$$

where A, B, C, D denote polynomial functions of the variable x. Those equations will be satisfied at the same time; for, it, as we proved, one will find:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Since, having changed x into $-x$, U becomes $-U$, but V is not changed, from the equations $(1.), (3.)$ the equations $(2.), (4.)$ follow immediately. To satisfy equations $(1.), (3.)$, $V + \lambda U$ must have two equal linear factors m times but $V + U$ must have two equal linear factors $m - 1$ times; in addition, $V + U$ must contain the factor $1 + x$. All this in total leads to $m + m - 1 + 1 = 2m$ conditional equations which is the number of unknowns $a, a', \cdots, a^{(m-1)}; b', b'', \cdots b^{(m)}$. Hence the propounded problem is well-defined.

Secondly, we want to prove that the transformation also succeeds having used a substitution of this kind:

$$y = \frac{x(a + a'x^2 + a''x^4 + a^{(m)}x^{2m})}{a + b'x^2 + b''x^4 + b^{(m)}x^{2m}} = \frac{U}{V},$$

which belongs to an odd number. Here, $V + U, V - U, V + \lambda U, V - \lambda U$ are all of odd order, whence we want to put:

$$\begin{aligned}
&(1.) \quad V + U &= (1 + x)AA \\
&(2.) \quad V - U &= (1 - x)BB \\
&(3.) \quad V + \lambda U &= (1 + kx)CC \\
&(4.) \quad V - \lambda U &= (1 - kx)DD.
\end{aligned}$$

Additionally, here only equations $(1.), (3.)$ have to be satisfied, from which by changing x into $-x$ the remaining two follow directly. To satisfy those equations, it is necessary that both $V + U$ and $V + \lambda U$ have two equal linear factors m-times, for which aim $2m$ conditional equations will have to be satisfied; additionally, $U + V$ needs to contain the factor $1 + x$. Hence we see that the number of conditional equations is $2m + 1$ which is the number of unknowns $a, a', a'', \cdots, a^{(m)}; b', b'', \cdots, b^{(m)}$. Therefore, the problem is also well-defined in this case.

11.

Let us denote polynomial functions of the variable y of such a kind by U', V' that, having put $z = \frac{U'}{V'}$, it is:

$$\frac{dz}{\sqrt{(1 - z^2)(1 - \mu^2 z^2)}} = \frac{dy}{\sqrt{(1 - y^2)(1 - \lambda^2 y^2)}}.$$

Let the substitution that was done, $z = \frac{U'}{V'}$, be of $p'-$th order and by means of another substitution $y = \frac{U}{V}$ (where U, V as above denote polynomial functions of the variable x) which we assume to be of order p, let us as above find:

$$\frac{dy}{\sqrt{(1 - y^2)(1 - \lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1 - x^2)(1 - k^2 x^2)}}.$$

Now, having substituted the value $y = \frac{U}{V}$ in the expression $z = \frac{U'}{V'}$, let $z = \frac{U''}{V''}$ be the result: Hence the one substitution $z = \frac{U''}{V''}$ by means of which it is found:

$$\frac{dz}{\sqrt{(1-z^2)(1-\mu^2 z^2)}} = \frac{dx}{MM'\sqrt{(1-x^2)(1-k^2x^2)}},$$

will be of order pp'. So, we see that from several successively used transformations belonging to the numbers $p, p', p'', \cdots$ one can compose another one belonging to the number $pp'p'' \cdots$. And, vice versa it is always possible - what we will not prove here - to compose a transformation which has the composite order $pp'p'' \cdots$ from other successively applied ones which have order $p, p', p'' \cdots$, respectively. Therefore, it is only necessary to investigate those transformations that belong to the *first* number from which all the others can be constructed. Now, in the following let us therefore put the first case aside which concerns the even order of transformation which can always be constructed from a transformation of odd order and a sufficiently often iterated transformation belonging to order 2. But let us examine the *second* case or the transformation of odd order in more detail now.

12.

We see that in this case we have to determine two functions, the one, V, of even order $2m$, the other, U, of odd order $2m+1$, in such a way that these equations hold:

$$V + U = (1+x)AA, \quad V + \lambda U = (1+kx)CC.$$

Now, I claim, if the functions U, V were determined in such a way that, having put $\frac{1}{kx}$ for x, $y = \frac{U}{V}$ becomes $\frac{1}{\lambda y} = \frac{V}{\lambda U}$, that then those equations follow from each other immediately.

Let us put $V = \varphi(x^2)$, $U = xF(x^2)$; we see that the expression $y = \frac{xF(x^2)}{\varphi(x^2)}$, having put $\frac{1}{kx}$ for x, goes over into

$$\frac{F\left(\frac{1}{k^2x^2}\right)}{kx\varphi\left(\frac{1}{k^2x^2}\right)} = \frac{x^{2m}F\left(\frac{1}{k^2x^2}\right)}{kx \cdot x^{2m}\varphi\left(\frac{1}{k^2x^2}\right)},$$

where $x^{2m}F\left(\frac{1}{k^2x^2}\right)$ and $x^{2m}\varphi\left(\frac{1}{k^2x^2}\right)$ are polynomial functions. To render this equal to the expression $\frac{1}{\lambda y} = \frac{V}{\lambda U} = \frac{\varphi(x^2)}{\lambda x F(x^2)}$, the following equations must hold

$$\varphi(x^2) = px^{2m}F\left(\frac{1}{k^2x^2}\right), \quad \lambda F(x^2) = pkx^{2m}\varphi\left(\frac{1}{k^2x^2}\right),$$

where p denotes a constant quantity. If we put $\frac{1}{kx}$ for x in these equation, we obtain: $\varphi\left(\frac{1}{k^2x^2}\right) = \frac{p}{k^{2m}x^{2m}}F(x^2)$ and $\lambda F\left(\frac{pk}{k^2x^2}\right) = \frac{pk}{k^{2m}x^{2m}}\varphi(x^2)$. Comparing these to the first equations we get $\frac{p}{k^{2m}} = \frac{\lambda}{pk}$, whence $p = \sqrt{\lambda k^{2m-1}}$. Therefore, it follows:

$$\varphi(x^2) = \sqrt{\lambda k^{2m-1}}x^{2m}F\left(\frac{1}{k^2x^2}\right), \quad F(x^2) = \sqrt{\frac{k^{2m+1}}{\lambda}}x^{2m}\varphi\left(\frac{1}{k^2x^2}\right);$$

The one of these equations follows from the other.

Now, as often as the expression

$$\frac{V+U}{1+x} = \frac{\varphi(x^2) + xF(x^2)}{1+x}$$

is a square of a polynomial function of the variable x, the same will also hold for the other equation which is derived from the first by putting $\frac{1}{kx}$ for x and multiplying by $\sqrt{\lambda k^{2m-1}}x^{2m}$. Having done this we obtain, *if* $\frac{V+U}{1+x}$ *is a square, that the function:*

$$\begin{aligned}\sqrt{\lambda k^{2m-1}}x^{2m}\frac{\varphi\left(\frac{1}{k^2x^2}\right) + \frac{1}{kx}F\left(\frac{1}{k^2x^2}\right)}{1+\frac{1}{kx}} &= \frac{\sqrt{\lambda k^{2m-1}}x^{2m}F\left(\frac{1}{k^2x^2}\right) + \sqrt{\lambda k^{2m+1}x^{2m+1}}\varphi\left(\frac{1}{k^2x^2}\right)}{1+kx}\\ &= \frac{\varphi(x^2) + \lambda xF(x^2)}{1+kx} = \frac{V+\lambda U}{1+kx}\end{aligned}$$

will itself be a square. Q.D.E.

Therefore, the problem was reduced to the other problem that the expression

$$\frac{\varphi(x^2) + \sqrt{\frac{k^{2m+1}}{\lambda}}x^{2m+1}\varphi\left(\frac{1}{k^2x^2}\right)}{1+x} = \frac{V+U}{1+x}$$

is made a square where $\varphi(x^2)$ denotes an expression of this kind:

$$\varphi(x^2) = V = 1 + b' + b''x^4 + \cdots + b^{(m)}x^{2m}.$$

But having put $U = xF(x^2) = x(a + a'x^2 + a''x^4 + \cdots + a^{(m)}x^{2m})$, because it is $U = xF(x^2) = \sqrt{\frac{k^{2m+1}}{\lambda}}x^{2m+1}\varphi\left(\frac{1}{k^2x^2}\right)$, we have:

$$(*) \qquad a = \sqrt{\frac{k}{\lambda}} \cdot \frac{b^{(m)}}{k^m}, \qquad a' = \sqrt{\frac{k}{\lambda}} \cdot \frac{b^{(m-1)}}{k^{m-2}}, \qquad a'' = \sqrt{\frac{k}{\lambda}} \cdot \frac{b^{(m-2)}}{k^{m-4}}, \cdots$$
$$a^{(m)} = \sqrt{\frac{k}{\lambda}} \cdot k^m \qquad a^{(m-1)} = \sqrt{\frac{k}{\lambda}} \cdot b'k^{m-2}, \quad a^{(m-2)} = \sqrt{\frac{k}{\lambda}} \cdot b''k^{m-4} \cdots$$

Now we will come to some examples.

1.5 A TRANSFORMATION OF THIRD ORDER IS PROPOUNDED

13.

Let $m = 1$ which is the simplest case, further let $V = 1 + b'x^2, U = x(a + a'x^2)$. Having put $A = 1 + \alpha x$ we find:

$$AA = (1 + \alpha x)^2 = 1 + 2\alpha x + \alpha\alpha x^2,$$

whence:

$$V + U = (1 + x)AA = 1 + (1 + 2\alpha)x + \alpha(2 + \alpha)x^2 + \alpha\alpha x^3.$$

Hence it is:

$$b' = \alpha(2 + \alpha), \quad a = (1 + 2\alpha), \quad a' = \alpha\alpha.$$

The equations $(*)$ in § 12 become the following:

$$a = \sqrt{\frac{k}{\lambda}} \cdot \frac{b'}{k}, \quad a' = \sqrt{\frac{k^3}{\lambda}},$$

whence we obtain:

$$1 + 2\alpha = \frac{\alpha(2 + \alpha)}{\sqrt{k\lambda}}, \quad \alpha\alpha = \sqrt{\frac{k^3}{\lambda}}, \quad \alpha = \sqrt[4]{\frac{k^3}{\lambda}}.$$

Put $\sqrt[4]{k} = u, \sqrt[4]{\lambda} = v$, it will be $\alpha = \frac{u^3}{v}$, $1 + 2\alpha = \frac{v+2u^3}{v}$, $\alpha(2 + \alpha) = \frac{u^3(2v+u^3)}{v^2}$. Therefore, the equation:

$$1 + 2\alpha = \frac{\alpha(2 + \alpha)}{\sqrt{k\lambda}}$$

goes over into the following:

$$\frac{v+2u^3}{v} = \frac{u(2v+u^3)}{v^4},$$

or

$$(1.) \qquad u^4 - v^4 + 2uv(1-u^2v^2) = 0.$$

Additionally, it is

$$\begin{aligned}
a \ &= 1+2\alpha = \frac{v+2u^3}{v} \\
a' &= \alpha\alpha = \frac{u^6}{v^2} \\
b' &= \alpha(2+\alpha) = \frac{u^3(2v+u^3)}{v^2} = vu^2(v+2u^3).
\end{aligned}$$

From this we obtain:

$$(2.) \qquad y = \frac{(v+2u^3)vx + u^6x^3}{v^2+v^3u^2(v+2u^3)x^2}.$$

Furthermore, because it is $1+y = \frac{(1+x)AA}{V}$, we obtain:

$$\begin{aligned}
&(3.) \qquad & 1+y \ &= \frac{(1+x)((v+u^3x)^2}{v^2+v^3u^2(v+2u^3)x^2} \\
&(4.) \qquad & 1-y \ &= \frac{(1-x)(v-u^3x)^2}{v^2+v^3u^2(v+2u^3)x^2} \\
&(5.) \qquad & \sqrt{\frac{1-y}{1+y}} &= \sqrt{\frac{1-x}{1+x}} \cdot \frac{v-u^3x}{v+u^3x} \\
&(6.) \qquad & \sqrt{1-y^2} &= \frac{\sqrt{1-x^2}(v^2-u^6x^2)}{v^2+v^3u^2(v+2u^3)x^2}.
\end{aligned}$$

Further, by putting $\frac{1}{kx} = \frac{1}{u^4x}$ for x, since y becomes $\frac{1}{\lambda y} = \frac{1}{v^4 y}$, we find the following system of equations:

$$(7.)\qquad 1+v^4y = \frac{(1+u^4x)(1+uvx)^2}{1+vu^2(v+2u^3)x^2}$$

$$(8.)\qquad 1-v^4y = \frac{(1-u^4x)(1-uvx)^2}{1+vu^2(v+2u^3)x^2}$$

$$(9.)\qquad \sqrt{\frac{1-v^4y}{1+v^y}} = \sqrt{\frac{1-u^4x}{1+u^4x}}\cdot\frac{1-uvx}{1+uvx}$$

$$(10.)\qquad \sqrt{1-v^8y^2} = \frac{\sqrt{1-u^8x^2}(1-u^2v^2x^2)}{1+vu^2(v+2u^3)x^2}.$$

14.

Having put

$$V+U=(1+x)AA,\quad V+\lambda U=(1+kx)CC,$$
$$V-U=(1-x)BB,\quad V-\lambda U=(1-kx)DD,$$

we see that it is:

$$ABCD = M\left(V\frac{dU}{dx}-U\frac{dV}{dx}\right),$$

where M denotes a constant quantity which can be found by comparison of the coefficients in the expressions $ABCD$, $V\frac{dU}{dx}-U\frac{dV}{dx}$. Now, having put $V=b+b'x^2$+etc., $U=ax+a'x^3$+etc. in the expressions A,B,C,D the constant term becomes $\sqrt{b}$, whence we see that in the product of all of them the constant becomes bb, but in the expression $V\frac{dU}{dx}-U\frac{dV}{dx}$ the constant becomes ab, whence:

$$M=\frac{b}{a}.$$

Therefore, in our example, because $b=1$, $a=\frac{v+2u^3}{v}=\frac{u(2v+u^3)}{v^4}$, it is:

$$M=\frac{v}{v+2u^3}=\frac{v^4}{u(2v+u^3)},$$

whence it follows:

$$\frac{dy}{\sqrt{(1-y^2)(1-v^8y^2)}} = \frac{v+2u^3}{v} \cdot \frac{dx}{\sqrt{(1-x^2)(1-u^8x^2)}}.$$

The moduli k, λ which we saw to depend on each other by means of an equation of fourth order in § 13 (1.) are easily expressed rationally by the same quantity α. For, from the formulas given above:

$$\alpha = \frac{u^3}{v}, \quad 1+2\alpha = \frac{\alpha(2+\alpha)}{\sqrt{k\lambda}} = \frac{\alpha(2+\alpha)}{u^2v^2}$$

it follows:

$$\alpha = \frac{u^3}{v}, \quad u^2v^2 = \frac{\alpha(2+\alpha)}{1+2\alpha},$$

whence it is:

$$u^8 = \frac{\alpha^3(2+\alpha)}{1+2\alpha} = k^2, \quad v^8 = \alpha\left(\frac{2+\alpha}{1+2\alpha}\right)^3 = \lambda^2.$$

Additionally, it is: $M = \frac{1}{1+2\alpha}$, whence having put $y = \sin T'$, $x = \sin T$, the equation:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$

becomes the following:

$$\frac{dT'}{\sqrt{(1+2\alpha)^3 - \alpha(2+\alpha)^3\sin^2 T'}} = \frac{dT}{\sqrt{(1+2\alpha - \alpha^3(2+\alpha)\sin^2 T}},$$

or this one:

$$\frac{dT'}{\sqrt{(1+2\alpha)^3\cos^2 T' + (1-\alpha)^3(1+\alpha)\sin^2 T'}} = \frac{dT}{\sqrt{(1+2\alpha)\cos^2 T + (1+\alpha)^3(1-\alpha)\sin^2 T}},$$

to which equation one gets by the substitution:

$$\sin T' = \frac{(1+2\alpha)\sin T + \alpha^2\sin^3 T}{1+\alpha(2+\alpha)\sin^2 T}.$$

15.

Now, let us treat the second simplest example in which $m = 2$,

$$V = 1 + b'x^2 + b''x^4, \quad U = x(a + a'x^2 + a''x^4), \quad A = \alpha x + \beta x^2.$$

We find:

$$AA = 1 + 2\alpha x + (2\beta + \alpha\alpha)x^2 + 2\alpha\beta x^3 + \beta\beta x^4,$$

whence it follows:

$$AA(1+x) = 1 + x(1+2\alpha) + x^2(2\alpha + 2\beta + \alpha\alpha) + x^3(2\beta + \alpha\alpha + 2\alpha\beta) + x^4(2\alpha\beta + \beta\beta) + \beta\beta x^5.$$

From this we obtain:

$$\begin{aligned} &b' = 2\alpha + 2\beta + \alpha\alpha, \quad b'' = \beta(2\alpha + \beta) \\ &a = 1 + 2\alpha, \quad a' = 2\beta + \alpha\alpha + 2\alpha\beta, \quad a'' = \beta\beta. \end{aligned}$$

The equations $(*)$ from § 12 become:

$$a = \sqrt{\frac{k}{\lambda}} \cdot \frac{b''}{k^2}, \quad a' = \sqrt{\frac{k}{\lambda}} \cdot b', \quad a'' = \sqrt{\frac{k^5}{\lambda}}.$$

From these it follows:

$$\frac{a'a'}{aa''} = \frac{b'b'}{b''},$$

or, because one has $b' = (2 + \alpha + \beta) + (\beta + \alpha\alpha)$, $a' = \beta(1 + 2\alpha) + (\beta + \alpha\alpha)$,

$$\frac{[(2\alpha + \beta) + (\beta + \alpha\alpha)]^2}{2\alpha + \beta} = \frac{[\beta(1 + 2\alpha) + (\beta + \alpha\alpha)]^2}{\beta(1 + 2\alpha)},$$

From this it easily follows:

$$\beta(1 + 2\alpha)(2\alpha + \beta) = (\beta + \alpha\alpha)^2,$$

which expanded and divided by α yields:

$$\alpha^3 = 2\beta(1+\alpha+\beta).$$

This equation can also be presented in these two ways:

$$\begin{aligned}(\alpha\alpha+\beta)(\alpha-2\beta) &= \beta(2\alpha)(1+2\alpha)\\ (\alpha\alpha+\beta)(2-\alpha) \quad &= (\alpha-2\beta)(2\alpha+\beta),\end{aligned}$$

whence it follows:

$$\left(\frac{2-\alpha}{\alpha-2\beta}\right)^2 = \frac{2\alpha+\beta}{\beta(1+2\alpha)}.$$

Having prepared these things the remaining are easily understood. For, having put $k = u^4$ and $\lambda = v^4$ we find:

$$\frac{2\alpha+\beta}{\beta(1+2\alpha)} = \frac{b''}{aa''} = \frac{b'b'}{a'a'} = \frac{\lambda}{k} = \frac{v^4}{u^4},$$

whence we also have:

$$\frac{2-\alpha}{\alpha-2\beta} = \frac{v^2}{u^2}.$$

Additionally, it is $\beta = \sqrt{a'} = \sqrt[4]{\frac{k^5}{\lambda}} = \frac{u^5}{v}$, whence the equations:

$$\frac{v^4}{u^4} = \left(\frac{2-\alpha}{\alpha-2\beta}\right)^2 = \frac{2\alpha+\beta}{\beta(1+2\alpha)}, \quad \frac{2-\alpha}{\alpha-2\beta} = \frac{v^2}{u^2}$$

become the following:

$$\begin{aligned}2\alpha v + u^5 &= uv^4(1+2\alpha)\\ u^2(2-\alpha) &= v(v\alpha - 2u^5)\end{aligned}$$

or:

$$\begin{aligned}2\alpha v(1-uv^3) &= u(v^4-u^4)\\ \alpha(v^2+u^2) &= 2u^2(1+u^3v),\end{aligned}$$

whence it is:

$$(u^2+v^2)(u^4-v^4)+4uv(1+u^3v)(1-uv^3)=0.$$

After the expansion it results:

$$(1.)\qquad u^6-v^6+5u^2v^2(u^2-v^2)+4uv(1-u^4v^4)=0.$$

The remaining things are found this way. From the equations:

$$\begin{aligned} 2\alpha v(1-uv^3) &= u(v^4-u^4) \\ \alpha(v^2+u^2) &= 2u^2(1+u^3v), \end{aligned}$$

it follows:

$$\alpha=\frac{u(v^4-u^4)}{2v(1-uv^3)}=\frac{2u^2(1+u^3v)}{u^2+v^2}.$$

From this it is:

$$\begin{aligned} a=1+2\alpha &= \frac{1}{v}\left(\frac{v-u^5}{1-uv^3}\right) \\ \beta+2\alpha=\frac{u^5}{v}+2\alpha &= uv^2\left(\frac{v-u^5}{1-uv^5}\right) \\ \alpha-2\beta=\alpha-\frac{2u^5}{v} &= \frac{2u^2}{v}\left(\frac{v-u^5}{u^2+v^2}\right) \\ 2-\alpha &= 2v\left(\frac{v-u^5}{u^2+v^2}\right) \\ \alpha\alpha+\beta=\frac{(\alpha-2\beta)(2\alpha-\beta)}{2-\alpha} &= u^3\left(\frac{v-u^5}{1-uv^3}\right). \end{aligned}$$

Finally, from this one deduces:

$$
\begin{aligned}
b' &= \beta + 2\alpha + \alpha\alpha + \beta = \frac{u(u^2+v^2)(v-u^5)}{1-uv^3} \\
b'' &= \frac{u^5}{v}(2\alpha+\beta) = u^6 v \left(\frac{v-u^5}{1-uv^5}\right) \\
a &= \frac{1}{v}\left(\frac{v-u^5}{1-uv^3}\right) \\
a' &= \frac{u^2}{v^2}\cdot b' = u^3\left(\frac{u^2+v^2}{v^2}\right)\left(\frac{v-u^5}{1-uv^3}\right) \\
a'' &= \frac{u^{10}}{v^2}.
\end{aligned}
$$

Now, because $M = \frac{1}{a} = v\left(\frac{1-uv^3}{v-u^5}\right)$, the transformation of fifth order will be contained in the following theorem:

Theorem

Having put:

$$
\begin{aligned}
&(1.) \quad u^6 - v^6 + 5uuv^2(u^2-v^2) + 4uv(1-u^4v^4) = 0 \\
&(2.) \quad y = \frac{v(v-u^5)x + u^3(u^2+v^2)(v-u^5)x^3 + u^{10}(1-uv^3)x^5}{v^2(1-uv^3) + uv^2(u^2+v^2)(v-u^5)x^2 + u^6v^3(v-u^5)x^4},
\end{aligned}
$$

it is:

$$
\frac{v(1-uv^3)dy}{\sqrt{(1-y^2)(1-v^8y^2)}} = \frac{(v-u^5)dx}{\sqrt{(1-x^2)(1-u^8x^2)}}.
$$

1.7 How to get to Multiplication by applying a Transformation twice

16.

Considering the equations between u and v found in the propounded examples:

$$
\begin{aligned}
&u^4 - v^4 + 2uv(1-u^2v^2) = 0 \\
&u^6 - v^6 + 5u^2v^2(u^2-v^2) + 4uv(1-u^4v^4) = 0,
\end{aligned}
$$

it is immediately seen that they remain unchanged if u is substituted for v and $-v$ for u. From this it follows from a theorem found in the first example having put:

$$u^4 - v^4 + 2uv(1 - u^2v^2) = 0$$
$$y = \frac{v(v + 2u^3)x + u^6x^3}{v^2 + v^3u^2(v + 2u^3)x^2},$$

that it is:

$$\frac{dy}{\sqrt{(1 - y^2)(1 - v^8y^2)}} = \frac{v + 2u^3}{v} \cdot \frac{dx}{\sqrt{(1 - x^2)(1 - u^8x^2)}},$$

on the other hand having put:

$$z = \frac{u(u - 2v^3)y + v^6y^3}{u^2 + u^3v^2(u - 2v^3)y^2}$$

it is immediately derived that it is:

$$\frac{dz}{\sqrt{(1 - z^2)(1 - u^8z^2)}} = \frac{u - 2v^3}{u} \cdot \frac{dy}{\sqrt{(1 - y^2)(1 - v^8y^2)}}.$$

But it is:

$$\left(\frac{v + 2u^3}{v}\right)\left(\frac{u - 2v^3}{u}\right) = \frac{2(u^4 - v^4) + uv(1 - 4u^2v^2)}{uv} = -3,$$

whence it follows:

$$\frac{dz}{\sqrt{(1 - z^2)(1 - u^8)}} = \frac{-3dx}{\sqrt{(1 - x^2)(1 - u^8x^2)}}.$$

To find 3 instead of -3, either z has to be changed to $-z$ or x to $-x$.

In like manner d from the theorem given in the second example having put:

$$z = \frac{u(u + v^5)y - v^3(u^2 + v^2)(u + v^5)y^3 + v^{10}(1 + u^3v)y^5}{u^2(1 + u^3v) - u^2v(u^2 + v^2)(u + v^5)y^2 + u^3v^6(u + v^5)y^4},$$

it is deduced that one finds:

$$\frac{dz}{\sqrt{(1 - z^2)(1 - u^8z^2)}} = \frac{u + v^5}{u(1 + u^3v)} \cdot \frac{dy}{\sqrt{(1 - y^2)(1 - v^8y^2)}}.$$

Now since from the equation:

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - uv)(1 - u^4v^4) = 0$$

it follows:

$$\frac{(u+v^5)(v-u^5)}{uv(1+u^3v)(1-uv^3)} = \frac{uv(1-u^4v^4) - (u^6-v^6)}{uv(1+u^3v)(1-uv^3)} = 5,$$

we see that:

$$\frac{dz}{\sqrt{(1-z^2)(1-u^8z^2)}} = \frac{5dx}{\sqrt{(1-x^2)(1-u^8x^2)}}.$$

So, by means of a twice applied transformation one reaches a multiplication.

These two examples, the transformations of third and fifth order, I at first exhibited in the letters I wrote to Schuhmacher in the month of June in the year 1827. See *Nova Astronomica* Nr. 123. And, at the same place I published the method by means of which they were found. On the other hand, they were already found by Legendre two years earlier.

1.8 On the new Notation of the elliptic Functions

17.

Having treated some algebraic questions we want to explore the analytic nature of our functions in more detail. Primarily, it is necessary to introduce a notation which will be useful in the following.

Having put $\int\limits_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = u$, the Geometers used to call the angle φ the *amplitude* of the function u. Therefore, we will denote this angle by $\operatorname{ampl} u$ in the following or in a shorter way by:

$$\varphi = \operatorname{am} u.$$

So, if

$$u = \int\limits_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

it will be:

$$x = \sin \operatorname{am} u.$$

Additionally, having put:

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = K,$$

we will call $K-u$ the complement of the function u; we will denote the amplitude of the complement by *coam* so that it is:

$$\operatorname{am}(K-u) = \operatorname{coam} u.$$

Following Legendre, we will denote the expression $\sqrt{1-k^2\sin^2\operatorname{am} u} = \frac{d\operatorname{am} u}{du}$ by:

$$\Delta \operatorname{am} u = \sqrt{1-k^2\sin^2\operatorname{am} u}.$$

I will denote the complement, as it was called by Legendre, of the modulus k by k' so that:

$$kk + k'k' = 1.$$

Further, in our notation it will be:

$$K' = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k'k'\sin^2\varphi}}.$$

The modulus which has to be kept in mind will be added either included in brackets or at the margin, if it is necessary. Not having added the modulus it is to be understood that the modulus is the same in all concerned formulas.

It is convenient to call the expressions $\sin\operatorname{am} u$, $\sin\operatorname{coam} u$, $\cos\operatorname{am}$, $\cos\operatorname{coam} u$, $\Delta\operatorname{am} u$, $\Delta\operatorname{coam} u$ etc. and *trigonometric functions of the amplitude elliptic functions* in the following so that we give that name another meaning than analysts have up to now. We will call u the *argument of the elliptic function* so that having put $x = \sin\operatorname{am} u$ it is $u = \arg\sin\operatorname{am} x$. In the notation we just introduced it will be:

$$\sin \operatorname{coam} u = \frac{\cos \operatorname{am} u}{\Delta \operatorname{am} u}$$
$$\cos \operatorname{coam} u = \frac{k' \sin \operatorname{am} u}{\Delta \operatorname{am} u}$$
$$\Delta \operatorname{coam} u = \frac{k'}{\Delta \operatorname{am} u}$$
$$\tan \operatorname{coam} u = \frac{1}{k' \tan \operatorname{am} u}$$
$$\cot \operatorname{coam} u = \frac{k'}{\cot \operatorname{am} u}.$$

1.9 Fundamental Formulas in the Analysis of elliptic Functions

18.

Let $\operatorname{am} u = a, \ \operatorname{am} u = b, \ \operatorname{am}(u+v) = \sigma, \ \operatorname{am}(u-v) = \vartheta$; the fundamental formulas for the addition and subtraction of elliptic functions are known:

$$\sin \sigma = \frac{\sin a \cos b \Delta b + \sin b \cos a \Delta a}{1 - k^2 \sin^2 a \sin^2 b}$$
$$\cos \sigma = \frac{\cos a \cos b - \sin a \sin b \Delta a \Delta b}{1 - k^2 \sin^2 a \sin^2 b}$$
$$\Delta \sigma = \frac{\Delta a \Delta b - k^2 \sin a \sin b \cos a \cos b}{1 - k^2 \sin^2 a \sin^2 b}$$
$$\sin \vartheta = \frac{\sin a \cos b \Delta b - \sin b \cos a \Delta a}{1 - k^2 \sin^2 a \sin^2 b}$$
$$\cos \vartheta = \frac{\cos a \cos b + \sin a \sin b \Delta a \Delta b}{1 - k^2 \sin^2 a \sin^2 b}$$
$$\Delta \vartheta = \frac{\Delta a \Delta b + k^2 \sin a \sin b \cos a \cos b}{1 - k^2 \sin^2 a \sin^2 b}.$$

To list all things which will be of use later we want to add the following formulas which are easily demonstrated and whose number is easily increased:

$$
\begin{aligned}
&(1.) && \sin\sigma + \sin\vartheta &&= \frac{2\sin a\cos b\Delta b}{1-k^2\sin^2 a\sin^2 b}\\
&(2.) && \cos\sigma + \cos\vartheta &&= \frac{2\cos a\cos b}{1-k^2\sin^2 a\sin^2 b}\\
&(3.) && \Delta\sigma + \Delta\vartheta &&= \frac{2\Delta a\Delta b}{1-k^2\sin^2 a\sin^2 b}\\
&(4.) && \sin\sigma - \sin\vartheta &&= \frac{2\sin b\cos a\Delta a}{1-k^2\sin^2 a\sin^2 b}\\
&(5.) && \cos\vartheta - \cos\sigma &&= \frac{2\sin a\sin b\Delta a\Delta b}{1-k^2\sin^2 a\sin^2 b}\\
&(6.) && \Delta\vartheta - \Delta\sigma &&= \frac{2k^2\sin a\cos b\cos a\cos b}{1-k^2\sin^2 a\sin^2 b}\\
&(7.) && \sin\sigma\sin\vartheta &&= \frac{\sin^2 a-\sin^2 b\Delta b}{1-k^2\sin^2 a\sin^2 b}\\
&(8.) && 1+k^2\sin\sigma\sin\vartheta &&= \frac{\Delta^2 b+k^2\sin^2 a\cos^2 b}{1-k^2\sin^2 a\sin^2 b}\\
&(9.) && 1+\sin\sigma\sin\vartheta &&= \frac{\cos^2 b+\sin^2 a\Delta^2 b}{1-k^2\sin^2 a\sin^2 b}\\
&(10.) && 1+\cos\sigma\cos\vartheta &&= \frac{\cos^2 a+\cos^2 b}{1-k^2\sin^2 a\sin^2 b}\\
&(11.) && 1+\Delta\sigma\Delta\vartheta &&= \frac{\Delta^2 a+\Delta^2 b}{1-k^2\sin^2 a\sin^2 b}\\
&(12.) && 1-k^2\sin\sigma\sin\vartheta &&= \frac{\Delta^2 a+k^2\sin^2 b\cos^2 a}{1-k^2\sin^2 a\sin^2 b}\\
&(13.) && 1-\sin\sigma\sin\vartheta &&= \frac{\cos^2 a+\sin^2 b\Delta^2 a}{1-k^2\sin^2 a\sin^2 b}\\
&(14.) && 1-\cos\sigma\cos\vartheta &&= \frac{\sin^2 a\Delta^2 b+\sin^2 b\Delta^2 a}{1-k^2\sin^2 a\sin^2 b}
\end{aligned}
$$

$$
\begin{aligned}
(15.)\quad & 1-\Delta\sigma\Delta\vartheta &&= \frac{k^2(\sin^2 a\cos^2 b+\sin^2\cos^2 a)}{1-k^2\sin^2 a\sin^2 b} \\
(16.)\quad & (1\pm\sin\sigma)(1\pm\sin\vartheta) &&= \frac{(\cos b\pm\sin a\Delta b)^2}{1-k^2\sin^2 a\sin^2 b} \\
(17.)\quad & (1\pm\sin\sigma)(1\mp\sin\vartheta) &&= \frac{(\cos a\pm\sin b\Delta a)^2}{1-k^2\sin^2 a\sin^2 b} \\
(18.)\quad & (1\pm\sin\sigma)(1\pm\sin\vartheta) &&= \frac{(\Delta b\pm k\sin a\cos b)^2}{1-k^2\sin^2 a\sin^2 b} \\
(19.)\quad & (1\pm\sin\sigma)(1\mp\sin\vartheta) &&= \frac{(\Delta a\pm k\sin b\cos a)^2}{1-k^2\sin^2 a\sin^2 b} \\
(20.)\quad & (1\pm\cos\sigma)(1\pm\cos\vartheta) &&= \frac{(\cos a\pm\cos b)^2}{1-k^2\sin^2 a\sin^2 b} \\
(21.)\quad & (1\pm\cos\sigma)(1\mp\cos\vartheta) &&= \frac{(\sin\Delta a\Delta b\mp\sin b\Delta a)^2}{1-k^2\sin^2 a\sin^2 b} \\
(22.)\quad & (1\pm\Delta\sigma)(1\pm\Delta\vartheta) &&= \frac{(\Delta a\pm\Delta b)^2}{1-k^2\sin^2 a\sin^2 b} \\
(23.)\quad & (1\pm\Delta\sigma)(1\mp\Delta\vartheta) &&= \frac{k^2\sin^2(a\mp b)}{1-k^2\sin^2 a\sin^2 b} \\
(24.)\quad & \sin\sigma\cos\vartheta &&= \frac{\sin a\cos a\Delta b+\sin b\cos b\Delta a}{1-k^2\sin^2 a\sin^2 b} \\
(25.)\quad & \sin\vartheta\cos\sigma &&= \frac{\sin a\cos a\Delta b-\sin b\cos b\Delta a}{1-k^2\sin^2 a\sin^2 b} \\
(26.)\quad & \sin\sigma\Delta\vartheta &&= \frac{\cos b\sin a\Delta a+\cos a\sin b\Delta b}{1-k^2\sin^2 a\sin^2 b} \\
(27.)\quad & \sin\vartheta\Delta\sigma &&= \frac{\cos b\sin a\Delta a-\cos a\sin b\Delta b}{1-k^2\sin^2 a\sin^2 b} \\
(28.)\quad & \cos\sigma\Delta\vartheta &&= \frac{\cos a\cos b\Delta a\Delta b-k'k'\sin a\sin b}{1-k^2\sin^2 a\sin^2 b} \\
(29.)\quad & \cos\vartheta\Delta\sigma &&= \frac{\cos a\cos b\Delta a\Delta b+k'k'\sin a\sin b}{1-k^2\sin^2 a\sin^2 b} \\
(30.)\quad & \sin(\sigma+\vartheta) &&= \frac{2\sin a\cos a\Delta b}{1-k^2\sin^2 a\sin^2 b}
\end{aligned}
$$

$$
\begin{aligned}
&(31.) \qquad \sin(\sigma-\vartheta) = \frac{2\sin b\cos b\Delta a}{1-k^2\sin^2 a\sin^2 b}\\
&(32.) \qquad \cos(\sigma+\vartheta) = \frac{\cos^2-\sin^2 a\Delta^2 b}{1-k^2\sin^2 a\sin^2 b}\\
&(33.) \qquad \cos(\sigma-\vartheta) = \frac{\cos^2 b-\sin^2\Delta^2 a}{1-k^2\sin^2 a\sin^2 b}.
\end{aligned}
$$

1.10 On imaginary Values of elliptic Functions. The Principle of double Periodicity

19.

Let us put $\sin\varphi = i\tan\psi$ where i is written for $\sqrt{-1}$ and more commonly used by geometers; it will be $\cos\varphi = \sec\psi = \frac{1}{\cos\psi}$ whence $d\varphi = \frac{id\psi}{\cos\psi}$. Therefore, it is:

$$\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{id\psi}{\sqrt{\cos^2\psi+k^2\sin^2\psi}} = \frac{id\psi}{\sqrt{1-k'k'\sin^2\psi}}.$$

We see that in our notation this equation goes over into this equation:

$$(1.) \qquad \sin\operatorname{am}(iu,k) = i\tan\operatorname{am}(u,k').$$

From this follows:

$$
\begin{aligned}
&(2.) \qquad \cos\operatorname{am}(iu,k) = \sec\operatorname{am}(u,k')\\
&(3.) \qquad \tan\operatorname{am}(ik,k) = i\sin\operatorname{am}(u,k')\\
&(4.) \qquad \Delta\operatorname{am}(iu,k) = \frac{\Delta\operatorname{am}(u,k')}{\cos\operatorname{am}(u,k')} = \frac{1}{\sin\operatorname{coam}(u,k')}\\
&(5.) \qquad \sin\operatorname{coam}(iu,k) = \frac{1}{\Delta\operatorname{am}(u,k')}\\
&(6.) \qquad \cos\operatorname{coam}(iu,k) = \frac{ik'}{k}\cos\operatorname{coam}(u,k')\\
&(7.) \qquad \tan\operatorname{coam}(iu,k) = \frac{-i}{k'\sin\operatorname{am}(u,k')}\\
&(8.) \qquad \Delta\operatorname{coam}(iu,k) = k'\sin\operatorname{coam}(u,k')
\end{aligned}
$$

Another system of formulas following from that one is this one:

$$
\begin{array}{lll}
(9.) & \sin\operatorname{am} 2iK' & = 0 \\
(10.) & \sin\operatorname{am} iK' & = \infty \quad \text{or if it pleases} \quad \pm i\infty \\
(11.) & \sin\operatorname{am}(u+2iK') & = \sin\operatorname{am} u \\
(12.) & \cos\operatorname{am}(u+2iK') & = -\cos\operatorname{am} u \\
(13.) & \Delta\operatorname{am}(u+2iK') & = -\Delta\operatorname{am} u \\
(14.) & \sin\operatorname{am}(u+iK') & = \dfrac{1}{k\sin\operatorname{am} u} \\
(15.) & \cos\operatorname{am}(u+iK') & = \dfrac{-i\Delta\operatorname{am} u}{k\sin\operatorname{am} u} = \dfrac{-ik'}{k\cos\operatorname{coam} u} \\
(16.) & \tan\operatorname{am}(u+iK') & = \dfrac{i}{\Delta\operatorname{am} u} \\
(17.) & \Delta\operatorname{am}(u+iK') & = -i\cot\sin\operatorname{am} u \\
(18.) & \sin\operatorname{coam}(u+iK') & = \dfrac{\Delta\operatorname{am} u}{k\cos\operatorname{am} u} = \dfrac{1}{k\sin\operatorname{coam} u} \\
(19.) & \cos\operatorname{coam}(u+iK') & = \dfrac{ik'}{k\cos\operatorname{am} u} \\
(20.) & \tan\operatorname{am}(u+iK') & = \dfrac{-i}{k'}\Delta\operatorname{am} u \\
(21.) & \Delta\operatorname{coam}(u+iK') & = ik'\sin\operatorname{am} u.
\end{array}
$$

From the preceding formulas which must be considered as fundamental formulas in the analysis of elliptic functions it is obvious that:

a) the elliptic functions of the imaginary argument iv and the modulus k can be transformed into another of the real argument v and modulus $k' = \sqrt{1-k^2}$. Therefore, in general it is possible to compose elliptic functions of the imaginary argument $u+iv$ and modulus k from elliptic functions of the argument u and modulus k and other of the argument v and the modulus k'.

b) the elliptic functions enjoy the property of double periodicity, one period being real, the other imaginary, if the modulus k is real. Both of them become imaginary, if the modulus itself is imaginary. We refer to this as *the principle of double periodicity*. From this, because it contains every possible periodicity, it is clear that elliptic functions might not be counted among other transcendental functions enjoying certain elegant properties, maybe even greater than those of the elliptic functions, but they have a certain kind of perfection.

1.11 Analytic Theory of the Transformation of elliptic Functions

20.

In the preceding paragraphs we saw that, if the polynomial functions of the variable x, A, B, C, D, U, V are determined in such a way that:

$$\begin{aligned} V + U &= (1+x)AA \\ V - U &= (1-x)BB \\ V + \lambda U &= (1+kx)CC \\ V - \lambda U &= (1-kx)DD, \end{aligned}$$

having put $y = \frac{U}{V}$ it will be:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$

while M denotes a constant quantity. Now, we want the general analytical expressions of those formulas.

Let n be an arbitrary odd integer, let m and m' be arbitrary positive or negative integers which nevertheless do not have a common factor which also divides the number n, let us put:

$$\omega = \frac{mK + m'iK'}{n};$$

then, the following equations hold:

$$
\begin{aligned}
U &= \frac{x}{M}\left(1-\frac{x^2}{\sin^2\operatorname{am}4\omega}\right)\left(1-\frac{x^2}{\sin^2\operatorname{am}8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{am}2(n-1)\omega}\right)\\
V &= (1-k^2x^2\sin^2\operatorname{am}4\omega)(1-k^2x^2\sin^2\operatorname{am}4\omega)\cdots(1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega)\\
A &= \left(1+\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1+\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1+\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\\
B &= \left(1-\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1-\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\\
C &= (1+kx\sin\operatorname{coam}4\omega)(1+kx\sin\operatorname{coam}8\omega)\cdots(1+kx\sin\operatorname{coam}2(n-1)\omega)\\
D &= (1-kx\sin\operatorname{coam}4\omega)(1-kx\sin\operatorname{coam}8\omega)\cdots(1-kx\sin\operatorname{coam}2(n-1)\omega)\\
\lambda &= k^n[\sin\operatorname{coam}4\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^4\\
M &= (-1)^{\frac{n-1}{2}}\left\{\frac{\sin\operatorname{coam}4\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}2(n-1)\omega}{\sin\operatorname{am}4\omega\sin\operatorname{am}4\omega\cdots\sin\operatorname{am}2(n-1)\omega}\right\}^2.
\end{aligned}
$$

Having constituted all this this, if $x=\sin\operatorname{am}u$, it is $y=\frac{U}{V}=\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)$.

Before we attempt the proof of the formulas itself we will indicate their transformation. For this purpose, we note the following formulas which are immediately deduced from the formulas in §. 18.

$$\text{(1.)}\qquad \sin\operatorname{am}(u+\alpha)\sin\operatorname{am}(u-\alpha) = \frac{\sin^2\operatorname{am}u-\sin^2\operatorname{am}\alpha}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}$$

$$\text{(2.)}\qquad \frac{[1+\sin\operatorname{am}(u+\alpha)][1+\sin\operatorname{am}(u-\alpha)]}{\cos^2\operatorname{am}\alpha} = \frac{\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\alpha}\right)^2}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}$$

$$\text{(3.)}\qquad \frac{[1-\sin\operatorname{am}(u+\alpha)][1-\sin\operatorname{am}(u-\alpha)]}{\cos^2\operatorname{am}\alpha} = \frac{\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\alpha}\right)^2}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}$$

$$\text{(4.)}\qquad \frac{[1+k\sin\operatorname{am}(u+\alpha)][1+k\sin\operatorname{am}(u-\alpha)]}{\Delta^2\operatorname{am}\alpha} = \frac{(1+k\sin\operatorname{am}u\sin\operatorname{coam}\alpha)^2}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}$$

$$\text{(5.)}\qquad \frac{[1-k\sin\operatorname{am}(u+\alpha)][1-k\sin\operatorname{am}(u-\alpha)]}{\Delta^2\operatorname{am}\alpha} = \frac{(1-k\sin\operatorname{am}u\sin\operatorname{coam}\alpha)^2}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}$$

From these formulas it also follows:

$$(6.)\qquad \frac{\cos\operatorname{am}(u+\alpha)\cos\operatorname{am}(u-\alpha)}{\cos^2\operatorname{am}\alpha} = \frac{1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{coam}\alpha}}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}$$

$$(7.)\qquad \frac{\Delta\operatorname{am}(u+\alpha)\Delta\operatorname{am}(u-\alpha)}{\Delta^2\operatorname{am}\alpha} = \frac{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{coam}\alpha}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}\alpha}.$$

Having put $x = \sin\operatorname{am}u$ from formula (1.) we obtain:

$$\frac{1-\frac{x^2}{\sin^2\operatorname{am}\alpha}}{1-k^2x^2\sin^2\operatorname{am}\alpha} = \frac{-\sin\operatorname{am}(u+\alpha)\sin\operatorname{am}(u-\alpha)}{\sin^2\operatorname{am}\alpha},$$

from the formulas (2.), (3.) we find:

$$\frac{\left(1\pm\frac{x}{\sin\operatorname{coam}\alpha}\right)^2}{1-k^2x^2\sin^2\operatorname{am}\alpha} = \frac{[1\pm\sin\operatorname{am}(u+\alpha)][1\pm\sin\operatorname{am}(u-\alpha)]}{\cos^2\operatorname{am}\alpha},$$

from the formulas (4.), (5.):

$$\frac{(1\pm kx\sin\operatorname{coam}\alpha)^2}{1-k^2x^2\sin^2\operatorname{am}\alpha} = \frac{[1\pm k\sin\operatorname{am}(u+\alpha)][1\pm k\sin\operatorname{am}(u-\alpha)]}{\Delta^2\operatorname{am}\alpha}.$$

Hence, if one successively puts $4\omega,\ 8\omega,\ \cdots, 2(n-1)\omega$ for α, but $4n\omega-\alpha$ for $-\alpha$, we will obtain:

$$(8.)\qquad \frac{U}{V} = \frac{\frac{x}{M}\left(1-\frac{x^2}{\sin^2\operatorname{am}4\omega}\right)\left(1-\frac{x^2}{\sin^2\operatorname{am}8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{am}2(n-1)\omega}\right)}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{\sin\operatorname{am}u\sin\operatorname{am}(u+4\omega)\sin\operatorname{am}(u+8\omega)\cdots\sin\operatorname{am}(u+4(n-1)\omega)}{[\sin\operatorname{coam}4\omega\sin\operatorname{coam}8\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^2}$$

$$(9.)\qquad \frac{(1+x)AA}{V} = \frac{(1+x)\left\{\left(1+\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1+\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1+\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1+\sin\operatorname{am}u][1+\sin\operatorname{am}(u+4\omega)][1+\sin\operatorname{am}(u+8\omega)]\cdots[1+\sin\operatorname{am}(u+4(n-1)\omega)]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2}$$

$$(10.)\qquad \frac{(1-x)BB}{V} = \frac{(1-x)\left\{\left(1-\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1-\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x}{\sin\operatorname{coam}4(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1-\sin\operatorname{am}u][1-\sin\operatorname{am}(u+4\omega)][1-\sin\operatorname{am}(u+8\omega)]\cdots[1-\sin\operatorname{am}(u+4(n-1)\omega)]}{[\cos\operatorname{am}4\omega][\cos\operatorname{am}8\omega]\cdots[\cos\operatorname{am}2(n-1)\omega]^2}$$

$$(11.)\quad \frac{(1+kx)CC}{V} = \frac{(1+kx)\left\{[1+kx\sin\operatorname{coam}4\omega][1+kx\sin\operatorname{coam}8\omega]\cdots[1+kx\sin\operatorname{coam}2(n-1)\omega]\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1+k\sin\operatorname{am}u][1+k\sin\operatorname{am}(u+4\omega)][1+k\sin\operatorname{am}(u+8\omega)]\cdots[1+k\sin\operatorname{am}(u+4(n-1)\omega)]}{[\Delta\operatorname{am}4\omega\Delta\operatorname{am}8\omega\cdots\Delta\operatorname{am}2(n-1)\omega]^2}$$

$$(12.)\quad \frac{(1-kx)DD}{V} = \frac{(1-kx)\left\{[1-kx\sin\operatorname{coam}4\omega][1-kx\sin\operatorname{coam}8\omega]\cdots[1-kx\sin\operatorname{coam}2(n-1)\omega]\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1-k\sin\operatorname{am}u][1-k\sin\operatorname{am}(u+4\omega)][1-k\sin\operatorname{am}(u+8\omega)]\cdots[1-k\sin\operatorname{am}(u+4(n-1)\omega)]}{[\Delta\operatorname{am}4\omega\Delta\operatorname{am}8\omega\cdots\Delta\operatorname{am}2(n-1)\omega]^2}$$

Hence also these formulas follow:

$$(13.)\quad \frac{\sqrt{1-x^2}AB}{V} = \sqrt{1-x^2}\,\frac{\left(1-\frac{x^2}{\sin^2\operatorname{coam}4\omega}\right)\left(1-\frac{x^2}{\sin^2\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{coam}2(n-1)\omega}\right)}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{\cos\operatorname{am}u\cos\operatorname{am}(u+4\omega)\cos\operatorname{am}(u+8\omega)\cdots\cos\operatorname{am}(u+4(n-1)\omega)}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2}$$

$$(14.)\quad \frac{\sqrt{1-k^2x^2}CD}{V} = \sqrt{1-x^2}\,\frac{[1-k^2x^2\sin^2\operatorname{coam}4\omega][1-k^2x^2\sin^2\operatorname{coam}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{coam}2(n-1)\omega]}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{\Delta\,\mathrm{u}\,\Delta\operatorname{am}(u+4\omega)\Delta\operatorname{am}(u+8\omega)\cdots\Delta\operatorname{am}(u+4(n-1)\omega)}{[\Delta\operatorname{am}4\omega\Delta\operatorname{am}8\omega\cdots\Delta\operatorname{am}2(n-1)\omega]^2}.$$

1.12 Proof of the analytic Formulas for the Transformation

21.

Now, let us demonstrate that having put:

$$1-y = (1-x)\frac{\left\{\left(1-\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1-\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$
$$= \frac{[1-\sin\operatorname{am}u][1-\sin\operatorname{am}(u+4\omega)][1-\sin\operatorname{am}(u+8\omega)]\cdots[1-\sin\operatorname{am}(u+4(n-1)\omega)]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2},$$

both the remaining formulas and this one are found:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}},$$

if:

$$\lambda \;=\; k^n[\sin\operatorname{coam}4\omega\sin\operatorname{coam}8\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^4$$
$$M=(-1)^{\frac{n-1}{2}}\frac{[\sin\operatorname{coam}4\omega\sin\operatorname{coam}8\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^2}{[\sin\operatorname{am}4\omega\sin\operatorname{am}8\omega\cdots\sin\operatorname{am}2(n-1)\omega]^2}.$$

From the propounded formula it is clear that y is not changed at all if u goes over into $u+4\omega$. For, then every factor will transform into the subsequent one but the last into the first. Hence y is generally not changed if $u+4p\omega$ is substituted for u where p denotes a negative or positive integer. On the other hand, if $u=0$, it is:

$$1-y=\frac{[1-\sin\operatorname{am}4\omega][1-\sin\operatorname{am}8\omega]\cdots[1-\sin\operatorname{am}4(n-1)\omega]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2}=1,$$

or $y=0$. For, it is easily seen that it is:

$$\begin{aligned}-\sin\operatorname{am}4(n-1)\omega&=\operatorname{am}4\omega\\ -\sin\operatorname{am}4(n-2)\omega&=\operatorname{am}8\omega\\ \cdots\cdots\cdots&\cdots\cdots ,\end{aligned}$$

whence

$$\begin{aligned}&[1-\sin\operatorname{am}4\omega][1-\sin\operatorname{am}4(n-1)\omega]=\cos^2\operatorname{am}4\omega\\ &[1-\sin\operatorname{am}8\omega][1-\sin\operatorname{am}4(n-2)\omega]=\cos^2\operatorname{am}4\omega\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &[1-\sin\operatorname{am}2(n-1)\omega][1-\sin\operatorname{am}4(n+1)\omega]=\cos^2\operatorname{am}2(n-1)\omega\end{aligned}$$

Now, because $y=0$, if $u=0$, and y is not changed, if $u+4p\omega$ is substituted for u, y vanishes in general, u takes on the following values:

$$0,\quad 4\omega,\quad 8\omega,\cdots\cdots,4(n-2)\omega,\quad 4(n-1)\omega,$$

to which the following values of the quantity $x=\sin\operatorname{am}u$ correspond:

$$0,\quad \sin\operatorname{am}4\omega,\quad \sin\operatorname{am}8\omega,\cdots,\sin\operatorname{am}4(n-2)\omega,\quad \sin\operatorname{am}4(n-1)\omega$$

which can also be exhibited this way:

$$0, \quad \pm \sin \operatorname{am} 4\omega, \quad \pm \sin \operatorname{am} 8\omega, \cdots, \pm \sin \operatorname{am} 2(n-1)\omega,$$

The values of the variable x it can have while y vanishes will all be different and their number will be n. Now, from the assumed equation between x and y from which we started, having put:

$$\begin{aligned} V &= [1 - k^2x^2 \sin^2 \operatorname{am} 4\omega][1 - k^2x^2 \sin^2 \operatorname{am} 8\omega] \cdots [1 - k^2x^2 \sin^2 \operatorname{am} 2(n-1)\omega] \\ &= [1 - k^2x^2 \sin^2 \operatorname{am} 2\omega][1 - k^2x^2 \sin^2 \operatorname{am} 4\omega] \cdots [1 - k^2x^2 \sin^2 \operatorname{am}(n-1)\omega], \end{aligned}$$

and $y = \frac{U}{V}$, it becomes obvious that U is a polynomial function of n-th order of the variable x. Because this functions vanishes together with y for the following n different values of the quantity x:

$$0, \quad \pm \sin \operatorname{am} 4\omega, \quad \pm \sin \operatorname{am} 8\omega, \cdots, \pm \sin \operatorname{am} 2(n-1)\omega,$$

it necessarily has the form:

$$\begin{aligned} U &= \frac{x}{M}\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 2\omega}\right)\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right) \cdots \left(1 - \frac{x^2}{\sin^2 \operatorname{am}(n-1)\omega}\right) \\ &= \frac{x}{M}\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 8\omega}\right) \cdots \left(1 - \frac{x^2}{\sin^2 \operatorname{am} 2(n-1)\omega}\right), \end{aligned}$$

where M denotes a constant. Because, having put $x = 1$, we have $1 - y = 0$ or $y = 1$, from the equation $y = \frac{U}{V}$ we obtain:

$$\begin{aligned} 1 &= \frac{\left(1 - \frac{1}{\sin^2 \operatorname{am} 2\omega}\right)\left(1 - \frac{1}{\sin^2 \operatorname{am} 4\omega}\right) \cdots \left(1 - \frac{1}{\sin^2 \operatorname{am}(n-1)\omega}\right)}{M[1 - k^2 \sin^2 \operatorname{am} 2\omega][1 - k^2 \sin^2 \operatorname{am} 4\omega] \cdots [1 - k^2 \sin^2 \operatorname{am}(n-1)2\omega]} \\ &= \frac{(-1)^{\frac{n-1}{2}} [\sin \operatorname{coam} 2\omega \sin \operatorname{coam} 4\omega \cdots \sin \operatorname{coam}(n-1)\omega]^2}{M[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2} \end{aligned}$$

whence it is

$$M = \frac{(-1)^{\frac{n-1}{2}} [\sin \operatorname{coam} 2\omega \sin \operatorname{coam} 4\omega \cdots \sin \operatorname{coam}(n-1)\omega]^2}{M[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2}.$$

There is a remarkable relation among the functions U, V, I mean the relation mentioned above, according to which it happens that, having put $\frac{1}{kx}$ for x, at the same time y goes to $\frac{1}{\lambda y}$ where λ denotes a constant.

For, having put $\frac{1}{kx}$ for x, the expression:

$$U = \frac{x}{M}\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 2\omega}\right)\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1 - \frac{x^2}{\sin^2 \operatorname{am}(n-1)\omega}\right)$$

goes over into this expression:

$$(-)^{\frac{n-1}{2}}\frac{V}{Mx^n}\cdot\frac{1}{k^n[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2}.$$

On the other hand by the same substitution

$$V = [1 - k^2x^2 \sin^2 \operatorname{am} 2\omega][1 - k^2x^2 \sin^2 \operatorname{am} 4\omega]\cdots[1 - k^2x^2 \sin^2 \operatorname{am} 2(n-1)\omega]$$

goes over into this expression:

$$(-1)^{\frac{n-1}{2}}\frac{U}{x^n}\cdot M[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am} n-1)\omega]^2.$$

Hence having replaced x by $\frac{1}{kx}$, $y = \frac{U}{V}$ goes over into:

$$\frac{U}{V}\cdot\frac{1}{MM\cdot k^n[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am} n-1)\omega]^2},$$

or y into $\frac{1}{\lambda y}$ if it is put:

$$\begin{aligned}\lambda &= MMk^n[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^4\\ &= k^n[\sin \operatorname{coam} 2\omega \sin \operatorname{coam} 4\omega \cdots \sin \operatorname{coam}(n-1)\omega]^4.\end{aligned}$$

This was to be proved.

From the propounded equation:

$$1 - y = (1-x)\frac{\left\{\left(1 - \frac{x}{\sin \operatorname{coam} 4\omega}\right)\left(1 - \frac{x}{\sin \operatorname{coam} 8\omega}\right)\cdots\left(1 - \frac{x}{\sin \operatorname{coam} 2(n-1)\omega}\right)\right\}^2}{[1 - k^2x^2 \sin^2 \operatorname{am} 4\omega][1 - k^2x^2 \sin^2 \operatorname{am} 8\omega]\cdots[1 - k^2x^2 \sin^2 \operatorname{am} 2(n-1)\omega]},$$

having put $\frac{1}{kx}$ for x, $\frac{1}{\lambda y}$ for y, what is possible from the preceding, we find:

$$\frac{1}{\lambda y} - 1 = \frac{1-kx}{\lambda U}\left\{[1 - kx\sin\operatorname{am}4\omega][1 - kx\sin\operatorname{am}8\omega]\cdots[1 - kx\sin\operatorname{am}2(n-1)\omega]\right\}^2$$

which multiplied by $\lambda y = \frac{\lambda U}{V}$ yields:

$$1 - \lambda y = (1-kx)\frac{\left\{[1 - kx\sin\operatorname{am}4\omega][1 - kx\sin\operatorname{am}8\omega]\cdots[1 - kx\sin\operatorname{am}2(n-1)\omega]\right\}^2}{V}.$$

Furthermore, it is clear that $y = \frac{U}{V}$ goes over into $-y$ if x is changed to $-x$ having done which we therefore immediately also obtain $1+y$, $1+\lambda y$ from $1-y$, $1-\lambda y$.

Therefore, we have now found polynomial functions U, V of the variable x of such a kind that it is

$$\begin{aligned}
V + U &= V(1+y) &&= (1+x)AA \\
V - U &= V(1-y) &&= (1-x)BB \\
V + \lambda U &= V(1+\lambda y) &&= (1+kx)CC \\
V - \lambda U &= V(1-\lambda y) &&= (1-kx)DD,
\end{aligned}$$

where A, B, C, D also denote polynomial functions of the variable x. But, from this according to the initially proved principles of the transformation it immediately follows:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2}}.$$

We obtain the multiplicator M, as we will call it from now on, from the observation made in § 14. Therefore, now all general analytical formulas concerning the theory of the transformation of elliptic functions are demonstrated.

22.

The propounded proof is derived from the one we gave in the *Nova Astronomica* Nr. 127 edited by Schuhmacher where ω is written instead of $\frac{K}{n}$, $(-1)^{\frac{n-1}{2}}M$ instead of M, while all other quantities are the same. First, I had communicated the general analytic theorem on the transformation in a slightly different form at the same place to the analysts in Nr. 123. Legendre, the greatest judge in this doctrine, wanted to review that proof ibidem in Nr. 130 in great detail. This in many ways venerable man observed that the equation:

$$V\frac{dU}{dx} - U\frac{dV}{dx} = \frac{ABCD}{M} = \frac{T}{M},$$

on which the proof is based and which in this treatise followed from principles of a mere algebraic transformation, can also be proved analytically without using those algebraic formulas. Because this remark of this remarkable man sheds much light on our theorem, let us demonstrate that equation in the same way as Legendre did, making less assumptions.

The propounded equation:

$$V\frac{dU}{dx} - U\frac{dV}{dx} = \frac{ABCD}{M} = \frac{T}{M}$$

can also exhibited this way:

$$\frac{dU}{Udx} - \frac{dV}{Vdx} = \frac{d\log U}{dx} - \frac{d\log V}{dx} = \frac{ABCD}{MUV} = \frac{T}{MUV}.$$

But we found:

$$U = \frac{x}{M}\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 2\omega}\right)\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1 - \frac{x^2}{\sin^2 \operatorname{am}(n-1)\omega}\right)$$
$$V = [1 - k^2x^2\sin^2 \operatorname{am} 2\omega][1 - k^2x^2\sin^2 \operatorname{am} 4\omega]\cdots[1 - k^2x^2\sin^2 \operatorname{am}(n-1)\omega],$$

whence it is:

$$\frac{d\log U}{dx} - \frac{d\log V}{dx} = \frac{1}{x} + \sum\left\{\frac{-2x}{\sin^2 \operatorname{am} 2q\omega} + \frac{2k^2x \operatorname{am} 2q\omega}{1 - k^2x^2\sin^2 \operatorname{am} 2q\omega}\right\},$$

after having assigned the values $1, 2, 3, \cdots, \frac{n-1}{2}$ to the number denoted by q. Furthermore, we found:

$$AB = \left(1 - \frac{x^2}{\sin^2 \operatorname{coam} 2\omega}\right)\left(1 - \frac{x^2}{\sin^2 \operatorname{coam} 4\omega}\right)\cdots\left(1 - \frac{x^2}{\sin^2 \operatorname{coam}(n-1)\omega}\right)$$
$$CD = [1 - k^2x^2\sin^2 \operatorname{coam} 2\omega][1 - k^2x^2\sin^2 \operatorname{coam} 4\omega]\cdots[1 - k^2x^2\sin^2 \operatorname{coam}(n-1)\omega],$$

whence it follows

$$\frac{T}{MUV} = \frac{ABCD}{MUV} = \frac{x\prod\left(1-\frac{x^2}{\sin^2\operatorname{coam}}2p\omega\right)(1-k^2x^2\sin^2\operatorname{coam}2p\omega)}{x^2\prod\left(1-\frac{x^2}{\sin^2\operatorname{am}}2p\omega\right)(1-k^2x^2\sin^2\operatorname{am}2p\omega)},$$

if in the products, for the sake of brevity denoted by the prefixed sign $\prod$, the values $1, 2, 3, \cdots \frac{n-1}{2}$ are assigned to the variable p. This expression can be decomposed into simple fractions such that it has this form:

$$\frac{1}{x} + \sum\left(\frac{A^{(q)}x}{\sin^2\operatorname{am}2q\omega} + \frac{B^{(q)}}{1-k^2x^2\sin^2\operatorname{am}2q\omega}\right);$$

Having done this, in order to reach what was propounded it must be demonstrated that it will be:

$$A^{(q)} = -2, \quad B^{(q)} = 2k^2\sin^2\operatorname{am}2q\omega$$

In the following we will denote the product formed in such a way that the values $1, 2, 3, \cdots, \frac{n-1}{2}$ are assigned to the element p by the prefixed sign $\prod^{(q)}$, but the value $p = q$ is omitted. Hence from the well-known theories of simple fractions it follows:

$$A^{(q)} = (1-k^2\sin^2\operatorname{am}q\omega\sin^2\operatorname{coam}2q\omega)\frac{\prod\left(\frac{1-\frac{\sin^2\operatorname{am}2q\omega}{\sin^2\operatorname{coam}2p\omega}}{1-k^2\sin^2\operatorname{am}2q\omega\sin^2\operatorname{am}2p\omega}\right)}{\prod^{(q)}\left(\frac{1-\frac{\sin^2\operatorname{am}2q\omega}{\sin^2\operatorname{am}2p\omega}}{1-k^2\sin^2\operatorname{am}2q\omega\sin^2\operatorname{coam}2p\omega}\right)}.$$

Now, from the formulas we exhibited above it is:

$$\frac{1-\frac{\sin^2\operatorname{am}2q\omega}{\sin^2\operatorname{coam}2p\omega}}{1-k^2\sin^2\operatorname{am}2q\omega\sin^2\operatorname{am}2p\omega} = \frac{\cos\operatorname{am}(2q+2p)\omega\cos\operatorname{am}(2p-2q)\omega}{\cos^2\operatorname{am}2p\omega}$$

$$\frac{1-\frac{\sin^2\operatorname{am}2q\omega}{\sin^2\operatorname{am}2p\omega}}{1-k^2\sin^2\operatorname{am}2q\omega\sin^2\operatorname{coam}2p\omega} = \frac{\cos\operatorname{coam}(2q+2p)\omega\cos\operatorname{coam}(2p-2q)\omega}{\cos^2\operatorname{coam}2p\omega}.$$

But having removed the factors which are the same in the denominator and numerator it easily seen to be:

$$\prod \frac{\cos \operatorname{am}(2q+2p)\omega \cos \operatorname{am}(2q-2p)\omega}{\cos^2 \operatorname{am} 2p\omega} = \frac{\pm 1}{\cos \operatorname{am} 2q\omega}$$

$$\prod^{(q)} \frac{\cos \operatorname{am}(2q+2p)\omega \cos \operatorname{am}(2q-2p)\omega}{\cos^2 \operatorname{am} 2p\omega} = \frac{\mp 1}{\cos \operatorname{am} 2q\omega} \cdot \frac{\cos^2 \operatorname{coam} 2q\omega}{\cos \operatorname{coam} 4p\omega} = \frac{\mp \cos \operatorname{coam} 2q\omega}{\cos \operatorname{coam} 4q\omega},$$

whence we find:

$$A^{(q)} = \frac{-(1-k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{coam} 2q\omega) \cos \operatorname{coam} 4q\omega}{\cos \operatorname{am} 2q\omega \cos \operatorname{coam} 2q\omega}.$$

But from the mentioned duplication formula it is:

$$\begin{aligned} \cos \operatorname{coam} 4q\omega &= \frac{2k' \sin \operatorname{am} 2q\omega \cos \operatorname{am} 2q\omega \Delta \operatorname{am} 2q\omega}{1 - 2k^2 \sin^2 \operatorname{am} 2q\omega + k^2 \sin^4 \operatorname{am} 2q\omega} \\ &= \frac{2k' \sin \operatorname{am} 2q\omega \cos \operatorname{am} 2q\omega \Delta \operatorname{am} 2q\omega}{\Delta^2 \operatorname{am} 2q\omega - k^2 \sin^2 2q\omega \cos^2 \operatorname{am} 2q\omega} \\ &= \frac{2 \cos \operatorname{coam} 2q\omega \cos \operatorname{coam} 2q\omega}{1 - k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{coam} 2q\omega}, \end{aligned}$$

whence finally, as it was to be demonstrated, it is $A^{(q)} = -2$. In like manner the other equation: $B^{(q)} = 2k^2 \sin^2 \operatorname{am} 2q\omega$ can be proved; this is nevertheless, already having found $A^{(q)} = -2$, achieved more easily the following way.

Having put $\frac{1}{kx}$ instead of x it is easily seen that the following expression is not changed:

$$\prod \frac{\left(1 - \frac{x^2}{\sin^2 \operatorname{coam} 2p\omega}\right)(1 - k^2 x^2 \sin^2 \operatorname{coam} 2p\omega)}{(1 - k^2 x^2 \sin^2 \operatorname{am} 2p\omega)\left(1 - \frac{x^2}{\sin^2 \operatorname{am} 2p\omega}\right)},$$

which expression can be put equal to the expression:

$$1 + \sum \frac{-2x^2}{\sin^2 \operatorname{am} 2q\omega - x^2} + \sum \frac{B^{(q)} x^2}{1 - k^2 x^2 \sin^2 \operatorname{am} 2q\omega}.$$

But this expression, having put $\frac{1}{kx}$ instead of x, goes over into this one:

$$1+\sum\frac{2}{1-k^2x^2\sin^2\operatorname{am}2q\omega}+\sum\frac{-B^{(q)}}{k^2(\sin^2\operatorname{am}2q\omega-x^2)}$$
$$=1+\sum\left(2-\frac{B^{(q)}}{k^2\sin^2\operatorname{am}2q\omega}\right)+\sum\frac{2k^2x^2\sin^2\operatorname{am}2q\omega}{1-k^2x^2\sin^2\operatorname{am}2q\omega}+\sum\frac{-B^{(q)}}{k^2\sin^2\operatorname{am}2q\omega}\cdots\frac{x^2}{\sin^2\operatorname{am}2q\omega-x^2},$$

whence for this expression to remain unchanged, what has to happen to complete the proof, it has to be:

$$B^{(q)}=2k^2\sin^2\operatorname{am}2q\omega.$$

Q.D.E.

23.

From formula (14.) in § 20 it follows:

$$\sqrt{1-\lambda^2y^2}=\sqrt{1-k^2x^2}\frac{CD}{V}$$
$$=\sqrt{1-k^2x^2}\frac{[1-k^2x^2\sin^2\operatorname{coam}2\omega][1-k^2x^2\sin^2\operatorname{coam}4\omega]\cdots[1-k^2x^2\sin^2\operatorname{coam}(n-1)\omega]}{[1-k^2x^2\sin^2\operatorname{am}2\omega][1-k^2x^2\sin^2\operatorname{am}4\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}(n-1)\omega]}.$$

Having put $x=1$, whence also $y=1$ and $\sqrt{1-\lambda^2}=\lambda'$, it is:

$$\lambda'=k'\left\{\frac{\Delta\operatorname{coam}2\omega\Delta\operatorname{coam}4\omega\cdots\Delta\operatorname{coam}(n-1)\omega}{\Delta\operatorname{am}2\omega\Delta\operatorname{am}4\omega\cdots\Delta\operatorname{am}(n-1)\omega}\right\}^2$$

But it is:

$$\Delta\operatorname{coam}=\frac{k'}{\operatorname{am}u},$$

whence it is:

$$(1.)\qquad \lambda'=\frac{k'^n}{[\Delta\operatorname{am}2\omega\Delta\operatorname{am}4\omega\cdots\Delta\operatorname{am}(n-1)\omega]^4}$$

Further, using the formulas:

$$(2.)\qquad \lambda=k^n[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}(n-1)\omega]^4$$
$$(3.)\qquad M=(-1)^{\frac{n-1}{2}}\frac{[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}(n-1)\omega]^2}{[\sin\operatorname{am}2\omega\sin\operatorname{am}4\omega\cdots\sin\operatorname{am}(n-1)\omega]^2},$$

we obtain:

$$
\begin{array}{lll}
(4.) & \dfrac{(-1)^{\frac{n-1}{2}}}{M}\sqrt{\dfrac{\lambda}{k^n}} & = [\sin\mathrm{am}\,2\omega\sin\mathrm{am}\,4\omega\cdots\sin\mathrm{am}(n-1)\omega]^2 \\
(5.) & \sqrt{\dfrac{\lambda k'^n}{\lambda' k^n}} & = [\cos\mathrm{am}\,2\omega\cos\mathrm{am}\,4\omega\cdots\cos\mathrm{am}(n-1)\omega]^2 \\
(6.) & \sqrt{\dfrac{k'^n}{\lambda'}} & = [\Delta\,\mathrm{am}\,2\omega\Delta\,\mathrm{am}\,4\omega\cdots\Delta\,\mathrm{am}(n-1)\omega]^2 \\
(7.) & \dfrac{(-1)^{\frac{n-1}{2}}}{M}\sqrt{\dfrac{\lambda'}{k'^n}} & = [\tan\mathrm{am}\,2\omega\tan\mathrm{am}\,4\omega\cdots\tan\mathrm{am}(n-1)\omega]^2 \\
(8.) & \sqrt{\dfrac{\lambda}{k^n}} & = [\sin\mathrm{coam}\,2\omega\sin\mathrm{coam}\,4\omega\cdots\sin\mathrm{coam}(n-1)\omega]^2 \\
(9.) & \dfrac{(-1)^{\frac{n-1}{2}}}{M}\sqrt{\dfrac{\lambda\lambda' k'^{n-2}}{k^n}} & = [\cos\mathrm{coam}\,2\omega\cos\mathrm{coam}\,4\omega\cdots\cos\mathrm{coam}(n-1)\omega]^2 \\
(10.) & \sqrt{\lambda' k'^{n-2}} & = [\Delta\,\mathrm{coam}\,2\omega\Delta\,\mathrm{coam}\,4\omega\cdots\Delta\,\mathrm{coam}(n-1)\omega]^2 \\
(11.) & (-1)^{\frac{n-1}{2}}M\sqrt{\dfrac{1}{\lambda' k'^{n-2}}} & = [\tan\mathrm{coam}\,2\omega\tan\mathrm{coam}\,4\omega\cdots\tan\mathrm{coam}(n-1)\omega]^2.
\end{array}
$$

By means of these formulas the formulas (8.), (13.), (14.) § 20 go over into the following

$$
\begin{array}{lll}
(12.) & \sin\mathrm{am}\left(\dfrac{u}{M},\lambda\right) & = \sqrt{\dfrac{k^n}{\lambda}}\sin\mathrm{am}\,u\sin\mathrm{am}(u+4\omega)\sin\mathrm{am}(u+8\omega)\cdots\sin\mathrm{am}(u+4(n-1)\omega) \\
(13.) & \cos\mathrm{am}\left(\dfrac{u}{M},\lambda\right) & = \sqrt{\dfrac{\lambda' k^n}{\lambda k'^n}}\cos\mathrm{am}\,u\cos\mathrm{am}(u+4\omega)\cos\mathrm{am}(u+8\omega)\cdots\cos\mathrm{am}(u+4(n-1)\omega) \\
(14.) & \Delta\,\mathrm{am}\left(\dfrac{u}{M},\lambda\right) & = \sqrt{\dfrac{\lambda'}{k'^n}}\Delta\,\mathrm{am}\,u\Delta\,\mathrm{am}(u+4\omega)\Delta\,\mathrm{am}(u+8\omega)\cdots\Delta\,\mathrm{am}(u+4(n-1)\omega)
\end{array}
$$

whence also:

$$
(15.)\quad \tan\mathrm{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{k'^n}{\lambda'}}\tan\mathrm{am}\,u\tan\mathrm{am}(u+4\omega)\tan\mathrm{am}(u+8\omega)\cdots\tan\mathrm{am}(u+4(n-1)\omega)
$$

So, another system of formulas is found. From equation (4.) it follows:

$$\frac{\lambda}{M^2k^n} = [\sin\operatorname{am} 2\omega \sin\operatorname{am} 4\omega \cdots \sin\operatorname{am}(n-1)\omega]^4,$$

whence it is:

$$y = \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right) = \frac{x}{M}\prod \frac{1 - \frac{x^2}{\sin^2 \sin\operatorname{am} 2p\omega}}{1 - k^2x^2\sin^2\operatorname{am} 2p\omega} = \frac{kM}{\lambda}x\prod\frac{x^2 - \sin^2\operatorname{am} 2p\omega}{x^2 - \frac{1}{k^2\sin^2\operatorname{am} 2p\omega}},$$

or:

$$0 = x\prod(x^2 - \sin^2\operatorname{am} 2p\omega) - \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)\prod\left(x^2 - \frac{1}{k^2\sin^2\operatorname{am} 2p\omega}\right).$$

The roots of this equation of n-th order are:

$$x = \sin\operatorname{am} u, \quad \sin\operatorname{am}(u+4\omega), \quad \sin\operatorname{am}(u+8\omega), \cdots, \sin\operatorname{am}(u+4(n-1)\omega),$$

whence we obtain the identity:

$$x\prod(x^2 - \sin^2\operatorname{am} 2p\omega) - \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)\prod\left(x^2 - \frac{1}{k^2\sin^2\operatorname{am} 2p\omega}\right)$$

$$= [x - \sin^2\operatorname{am} u][x - \sin^2\operatorname{am}(u+4\omega)][x - \sin^2\operatorname{am}(u+8\omega)]\cdots[x - \sin^2\operatorname{am}(u+4(n-1)\omega)].$$

From this the sum of roots results as:

$$(16.) \quad \sum\sin(u+4q\omega) = \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right).$$

In the same way it is found

$$(17.) \quad \sum\cos\operatorname{am}(u+4q\omega) = \frac{(-1)^{\frac{n-1}{2}}\lambda}{kM}\cos\operatorname{am}\left(\frac{u}{M}, \lambda\right)$$

$$(18.) \quad \sum\Delta\operatorname{am}(u+4q\omega) = \frac{(-1)^{\frac{n-1}{2}}\lambda}{M}\Delta\operatorname{am}\left(\frac{u}{M}, \lambda\right)$$

$$(19.) \quad \sum\tan\operatorname{am}(u+4q\omega) = \frac{\lambda'}{k'M}\tan\operatorname{am}\left(\frac{u}{M}, \lambda\right),$$

in which formula the values $0, 1, 2, 3, \cdots, n-1$ are assigned to the number q. It is convenient to represent also these formulas in this way:

$$\begin{aligned}
\frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \sin\operatorname{am} + \sum[\sin\operatorname{am}(u+4q\omega)+\sin\operatorname{am}(u-4q\omega)]\\
\frac{(-1)^{\frac{n-1}{2}}\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \cos\operatorname{am} + \sum[\cos\operatorname{am}(u+4q\omega)+\cos\operatorname{am}(u-4q\omega)]\\
\frac{(-1)^{\frac{n-1}{2}}\lambda}{M}\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \Delta\operatorname{am} \quad + \sum[\Delta\operatorname{am}(u+4q\omega)+\Delta\operatorname{am}(u-4q\omega)]\\
\frac{\lambda'}{k'M}\tan\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \tan\operatorname{am} + \sum[\tan\operatorname{am}(u+4q\omega)+\tan\operatorname{am}(u-4q\omega)],
\end{aligned}$$

where the number q takes on the values $1, 2, 3, \cdots, \frac{n-1}{2}$. Now, let us note the following formulas:

$$\begin{aligned}
\sin\operatorname{am}(u+4q\omega)+\sin\operatorname{am}(u-4q\omega) &= \frac{2\cos\operatorname{am}4q\omega\Delta\operatorname{am}4q\omega\sin\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
\cos\operatorname{am}(u+4q\omega)+\cos\operatorname{am}(u-4q\omega) &= \frac{2\cos\operatorname{am}4q\omega\cos\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
\Delta\operatorname{am}(u+4q\omega)\quad+\Delta\operatorname{am}(u-4q\omega) &= \frac{2\Delta\operatorname{am}4q\omega\Delta\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
\tan\operatorname{am}(u+4q\omega)+\tan\operatorname{am}(u-4q\omega) &= \frac{2\Delta\operatorname{am}4q\omega\sin\operatorname{am}u\cos\operatorname{am}u}{\cos^2\operatorname{am}4q\omega-\Delta^2\operatorname{am}4q\omega\sin^2\operatorname{am}u},
\end{aligned}$$

using which the formulas (16.) – (19.) go over into these:

$$\begin{aligned}
&(20.) & \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \sin\operatorname{am}u + \sum\frac{2\cos\operatorname{am}4q\omega\Delta\operatorname{am}4q\omega\sin\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
&(21.) & \frac{(-1)^{\frac{n-1}{2}}\lambda}{kM}\cos\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \cos\operatorname{am}u + \sum\frac{2\cos\operatorname{am}4q\omega\cos\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
&(22.) & \frac{(-1)^{\frac{n-1}{2}}\lambda}{M}\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \Delta\operatorname{am} \quad + \sum\frac{2\Delta\operatorname{am}4q\omega\Delta\operatorname{am}u}{1-k^2\sin^2\operatorname{am}4q\omega\sin^2\operatorname{am}u}\\
&(23.) & \frac{\lambda'}{k'M}\tan\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \tan\operatorname{am}u + \sum\frac{2\Delta\operatorname{am}4q\omega\sin\operatorname{am}u\cos\operatorname{am}u}{\cos^2\operatorname{am}4q\omega-\Delta^2\operatorname{am}4q\omega\sin^2\operatorname{am}u},
\end{aligned}$$

which are also obtained, if the formulas propounded above are resolved into simple fractions by known methods.

1.13 On various Transformations of the same Order. Two real Transformations, of a larger Modulus into a smaller and of smaller into a larger

24.

We saw that we can assign an arbitrary value of the form $\frac{mK+m'iK}{n}$ to the variable ω while m and m' denote positive or negative integer numbers which nevertheless, if n is a composite number, do not have a factor of n in common. But it easily seen, if q is a prime number, that the values $\frac{qmK+im'iK'}{n}$ will not exhibit different substitutions. Hence, if n itself is a prime number, all values of the variable ω which yield different transformations will be:

$$\frac{K}{n}, \quad \frac{iK'}{n}, \quad \frac{K+iK'}{n}, \quad \frac{K+2iK'}{n}, \quad \frac{K+3iK'}{n}, \quad \cdots \quad \frac{K+(n-1)iK'}{n},$$

or also:

$$\frac{K}{n}, \quad \frac{iK'}{n}, \quad \frac{K+iK'}{n}, \quad \frac{2K+iK'}{n}, \quad \frac{3K+iK'}{n}, \dots, \frac{(n-1)K+iK'}{n},$$

or, if it pleases:

$$\frac{K}{n}, \quad \frac{iK'}{n}, \quad \frac{K\pm iK'}{n}, \quad \frac{K\pm 2iK'}{n}, \quad \frac{K\pm 3iK'}{n}, \dots, \frac{K\pm\frac{n-1}{2}iK'}{n},$$

or also:

$$\frac{K}{n}, \quad \frac{iK'}{n}, \quad \frac{K\pm iK'}{n}, \quad \frac{2K\pm iK'}{n}, \quad \frac{3K\pm iK'}{n}, \dots, \frac{\frac{n-1}{2}K\pm iK'}{n},$$

whose total amount is $n+1$. And indeed we saw in the transformations of third and fifth order propounded above as examples that the equations among $u=\sqrt[4]{k}$ and $\sqrt[4]{v}$, which we called *modular equations*, ascended to fourth and sixth degree, respectively. But if n is a composite number, this number is vastly increased; for, the cases in which either m or m' or even both have a certain common factor, but the common factor of m, m' is not a factor of n, are additionally to be included. In general, the following theorem holds:

"The number of mutually different substitutions of n-th order by means of which it is possible to transform elliptic functions is equal to the sum of factors of n which number nevertheless, if n is not square-free, contains substitutions mixed from a transformation and multiplication, and hence, if n is a perfect square, contains the multiplication itself."

Therefore, this sum of factors will denote the degree to which for given number n the modular equation will ascend, where it is to be noted, if n is a square, that one of the total number of roots will yield $k = \lambda$, and generally that, if $n = mv^2$, while m^2 denotes a mixed square dividing n, all of the total number of roots also will be roots of the modular equation which belongs to v itself.

Among the values of the variable ω propounded above which in the case of a prime n, which case, since the remaining case reduce to it, is convenient to consider it separately, yielded the total amount of transformations, generally, only two are found which yield real transformations; namely $\omega = \frac{K}{n}$, $\omega' = \frac{iK'}{n}$. In the following, we will call the one case the *first* transformation, the other the *second*. And we will denote the moduli corresponding to them by λ, λ_1, respectively, and their complements by λ', λ_1'. We will denote the arguments of the amplitude $\frac{\pi}{2}$ corresponding to these moduli (Legendre calls them complete elliptic integrals) by Λ, Λ_1, Λ', Λ_1'. Our general formulas for these two case are the following.

I.

Formulas for the first real Transformations of the Modulus k into the Modulus λ.

$$\begin{aligned}
\lambda &= k^n \left\{ \sin\operatorname{coam}\frac{2K}{n}\sin\operatorname{coam}\frac{4K}{n}\cdots\sin\operatorname{coam}\frac{(n-1)K}{n}\right\}^4 \\
\lambda' &= \frac{k'^n}{\left\{\Delta\operatorname{coam}\frac{2K}{n}\Delta\operatorname{coam}\frac{4K}{n}\cdots\Delta\operatorname{coam}\frac{(n-1)K}{n}\right\}^4} \\
M &= \left\{\frac{\sin\operatorname{coam}\frac{2K}{n}\sin\operatorname{coam}\frac{4K}{n}\cdots\sin\operatorname{coam}\frac{(n-1)K}{n}}{\sin\operatorname{am}\frac{2K}{n}\sin\operatorname{am}\frac{4K}{n}\cdots\sin\operatorname{am}\frac{(n-1)K}{n}}\right\}^2
\end{aligned}$$

$$\begin{aligned}
\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \frac{\frac{\sin\operatorname{am}u}{M}\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{am}\frac{2K}{n}}\right)\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{am}\frac{4K}{n}}\right)\cdots\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{am}\frac{(n-1)K}{n}}\right)}{\left(1-k^2\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am}u\right)\left(1-k^2\sin^2\operatorname{am}\frac{4K}{n}\sin^2\operatorname{am}u\right)\cdots\left(1-k^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\sin^2\operatorname{am}u\right)} \\
&= (-1)^{\frac{n-1}{2}}\sqrt{\frac{k^n}{\lambda}}\sin\operatorname{am}u\sin\operatorname{am}\left(u+\frac{4K}{n}\right)\sin\operatorname{am}\left(u+\frac{8K}{n}\right)\cdots\sin\operatorname{am}\left(u+\frac{4(n-1)K}{n}\right) \\
\cos\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \frac{\cos\operatorname{am}u\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{coam}\frac{2K}{n}}\right)\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{coam}\frac{4K}{n}}\right)\cdots\left(1-\frac{\sin^2\operatorname{am}u}{\sin^2\operatorname{coam}\frac{(n-1)K}{n}}\right)}{\left(1-k^2\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am}u\right)\left(1-k^2\sin^2\operatorname{am}\frac{4K}{n}\sin^2\operatorname{am}u\right)\cdots\left(1-k^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\sin^2\operatorname{am}u\right)} \\
&= \sqrt{\frac{\lambda' k^n}{\lambda k'^n}}\cos\operatorname{am}u\cos\operatorname{am}\left(u+\frac{4K}{n}\right)\cos\operatorname{am}\left(u+\frac{8K}{n}\right)\cdots\cos\operatorname{am}\left(u+\frac{4(n-1)K}{n}\right) \\
\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \frac{\Delta\operatorname{am}\left(1-k^2\sin^2\operatorname{coam}\frac{2K}{n}\sin^2\operatorname{am}u\right)\left(1-k^2\sin^2\operatorname{coam}\frac{4K}{n}\sin^2\operatorname{am}u\right)\cdots\left(1-k^2\sin^2\operatorname{coam}\frac{(n-1)K}{n}\sin^2\operatorname{am}u\right)}{\left(1-k^2\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am}u\right)\left(1-k^2\sin^2\operatorname{am}\frac{4K}{n}\sin^2\operatorname{am}u\right)\cdots\left(1-k^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\sin^2\operatorname{am}u\right)} \\
&= \sqrt{\frac{\lambda'}{k'^n}}\Delta\operatorname{am}u\Delta\operatorname{am}\left(u+\frac{4K}{n}\right)\Delta\operatorname{am}\left(u+\frac{8K}{n}\right)\cdots\Delta\operatorname{am}\left(u+\frac{4(n-1)K}{n}\right)
\end{aligned}$$

$$\sqrt{\frac{1\mp\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}{1\pm\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}}$$

$$=\sqrt{\frac{1-\sin\operatorname{am}u}{1+\sin\operatorname{am}u}}\cdot\frac{\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{4K}{n}}\right)\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{8K}{n}}\right)\cdots\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{2(n-1)K}{n}}\right)}{\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{4K}{n}}\right)\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{8K}{n}}\right)\cdots\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{2(n-1)K}{n}}\right)}$$

$$\sqrt{\frac{1\mp\lambda\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}{1\pm\lambda\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}}$$

$$=\sqrt{\frac{1-k\sin\operatorname{am}u}{1+k\sin\operatorname{am}u}}\cdot\frac{\left(1-k\sin\operatorname{coam}\frac{4K}{n}\sin\operatorname{am}u\right)\left(1-k\sin\operatorname{coam}\frac{8K}{n}\sin\operatorname{am}u\right)\cdots\left(1-k\sin\operatorname{coam}\frac{2(n-1)K}{n}\sin\operatorname{am}u\right)}{\left(1+k\sin\operatorname{coam}\frac{4K}{n}\sin\operatorname{am}u\right)\left(1+k\sin\operatorname{coam}\frac{8K}{n}\sin\operatorname{am}u\right)\cdots\left(1+k\sin\operatorname{coam}\frac{2(n-1)K}{n}\sin\operatorname{am}u\right)}$$

$$
\begin{aligned}
\frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \sin\operatorname{am} u + 2\sum\frac{(-1)^q\cos\operatorname{am}\frac{2qK}{n}\Delta\operatorname{am}\frac{2qK}{n}\sin\operatorname{am} u}{1-k^2\sin^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am} u}\\
\frac{\lambda}{kM}\cos\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \cos\operatorname{am} u + 2\sum\frac{(-1)^q\cos\operatorname{am}\frac{2qK}{n}\cos\operatorname{am} u}{1-k^2\sin^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am} u}\\
\frac{1}{M}\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \Delta\operatorname{am} u + 2\sum\frac{\Delta\operatorname{am}\frac{2qK}{n}\Delta\operatorname{am} u}{1-k^2\sin^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am} u}\\
\frac{\lambda'}{k'M}\tan\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \tan\operatorname{am} u + 2\sum\frac{\tan\operatorname{am}\frac{2qK}{n}\sin\operatorname{am} u\cos\operatorname{am} u}{\cos^2\operatorname{am}\frac{2qK}{n}-\Delta^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am} u}.
\end{aligned}
$$

II.

A. Formulas for the second real Transformation, of the Modulus k into the Modulus λ_1 under an imaginary Form

$$\lambda_1 = k^n \left\{ \sin\operatorname{coam} \frac{2iK'}{n} \sin\operatorname{coam} \frac{4iK'}{n} \cdots \sin\operatorname{coam} \frac{(n-1)K}{n} \right\}^4$$

$$\lambda_1' = \frac{k'^n}{\left\{ \Delta\operatorname{coam} \frac{2iK'}{n} \Delta\operatorname{coam} \frac{4iK'}{n} \cdots \Delta\operatorname{coam} \frac{(n-1)iK'}{n} \right\}^4}$$

$$M = (-1)^{\frac{n-1}{2}} \left\{ \frac{\sin\operatorname{coam} \frac{2iK'}{n} \sin\operatorname{coam} \frac{4iK'}{n} \cdots \sin\operatorname{coam} \frac{(n-1)iK'}{n}}{\sin\operatorname{am} \frac{2iK'}{n} \sin\operatorname{am} \frac{4iK'}{n} \cdots \sin\operatorname{am} \frac{(n-1)iK'}{n}} \right\}^2$$

$$\sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right) = \frac{\frac{\sin\operatorname{am} u}{M_1}\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{2iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{4iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{(n-1)iK'}{n}}\right)}{\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{(n-2)iK'}{n}}\right)}$$

$$= \sqrt{\frac{k^n}{\lambda_1}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4iK'}{n}\right) \sin\operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots \sin\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right)$$

$$\cos\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right) = \frac{\cos\operatorname{am} u \left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{2iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{4iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{(n-1)iK'}{n}}\right)}{\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{(n-2)iK'}{n}}\right)}$$

$$= \sqrt{\frac{\lambda_1' k^n}{\lambda_1 k'^n}} \cos\operatorname{am} u \cos\operatorname{am}\left(u + \frac{4iK'}{n}\right) \cos\operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots \cos\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right)$$

$$\Delta\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right) = \frac{\Delta\operatorname{am} u \left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam} \frac{(n-2)iK'}{n}}\right)}{\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am} \frac{(n-2)iK'}{n}}\right)}$$

$$= \sqrt{\frac{\lambda_1'}{k'^n}} \Delta\operatorname{am} u \Delta\operatorname{am}\left(u + \frac{4iK'}{n}\right) \Delta\operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots \Delta\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right)$$

$$\sqrt{\frac{1 - \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right)}{1 + \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right)}}$$

$$= \sqrt{\frac{1 - \sin\operatorname{am} u}{1 + \sin\operatorname{am} u}} \cdot \frac{\left(1 - \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{2iK'}{n}}\right)\left(1 - \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{4iK'}{n}}\right)\cdots\left(1 - \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{(n-1)iK'}{n}}\right)}{\left(1 + \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{2iK'}{n}}\right)\left(1 + \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{4iK'}{n}}\right)\cdots\left(1 + \frac{\sin\operatorname{am} u}{\sin\operatorname{coam} \frac{(n-1)iK'}{n}}\right)}$$

$$\sqrt{\frac{1 - \lambda_1 \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right)}{1 + \lambda_1 \sin\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right)}}$$

$$= \sqrt{\frac{1-k\sin\mathrm{am}\,u}{1+k\sin\mathrm{am}\,u}} \cdot \frac{\left(1-\frac{\sin\mathrm{am}\,u}{\sin\mathrm{coam}\,\frac{iK'}{n}}\right)\left(1-\frac{\sin\mathrm{am}\,u}{\sin\mathrm{coam}\,\frac{3iK'}{n}}\right)\cdots\left(1-\frac{\sin\mathrm{am}\,u}{\sin\mathrm{coam}\,\frac{(n-2)iK'}{n}}\right)}{\left(1+\frac{\sin\mathrm{am}\,u}{\sin\mathrm{coam}\,\frac{iK'}{n}}\right)\left(1+\frac{\sin\mathrm{am}\,u}{\sin\mathrm{coam}\,\frac{3iK'}{n}}\right)\cdots\left(1+\frac{\sin\mathrm{am}\,u}{\sin\mathrm{coam}\,\frac{(n-2)iK'}{n}}\right)}$$

$$\begin{aligned}
\frac{\lambda_1}{kM_1}\sin\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) &= \sin\mathrm{am}\,u + \frac{2}{k}\sum\frac{\cos\mathrm{am}\,\frac{(2q-1)iK'}{n}\,\Delta\,\mathrm{am}\,\frac{(2q-1)iK'}{n}\,\sin\mathrm{am}\,u}{\sin^2\mathrm{am}\,\frac{(2q-1)iK'}{n}-\sin^2\mathrm{am}\,u} \\
\frac{(-1)^{\frac{n-1}{2}}\lambda_1}{kM_1}\cos\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) &= \cos\mathrm{am}\,u + \frac{2(-1)^{\frac{n-1}{2}}}{ik}\sum\frac{(-1)^q\sin\mathrm{am}\,\frac{(2q-1)iK'}{n}\,\Delta\,\mathrm{am}\,\frac{(2q-1)iK'}{n}\,\cos\mathrm{am}\,u}{\sin^2\mathrm{am}\,\frac{(2q-1)iK'}{n}-\sin^2\mathrm{am}\,u} \\
\frac{(-1)^{\frac{n-1}{2}}}{M_1}\Delta\,\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) &= \Delta\,\mathrm{am}\,u \quad + \frac{2(-1)^{\frac{n-1}{2}}}{i}\sum\frac{(-1)^q\sin\mathrm{am}\,\frac{(2q-1)iK'}{n}\,\Delta\,\mathrm{am}\,\frac{(2q-1)iK'}{n}\,\cos\mathrm{am}\,u}{\sin^2\mathrm{am}\,\frac{(2q-1)iK'}{n}-\sin^2\mathrm{am}\,u} \\
\frac{\lambda_1'}{k'M_1}\tan\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right) &= \tan\mathrm{am}\,u + 2\sum\frac{(-1)^q\Delta\,\mathrm{am}\,\frac{2qiK'}{n}\,\sin\mathrm{am}\,u\cos\mathrm{am}\,u}{\cos^2\mathrm{am}\,\frac{2qiK'}{n}-\Delta^2\,\mathrm{am}\,\frac{2qiK'}{n}\,\sin^2\mathrm{am}\,u}
\end{aligned}$$

II.

B. Formulas for the second real Transformation in real Form

$$\lambda_1 = \frac{k^n}{\left\{\Delta\operatorname{am}\left(\frac{2K'}{n},k'\right)\left(\frac{4K'}{n},k'\right)\cdots\left(\frac{(n-1)K'}{n},k'\right)\right\}^4}$$

$$\lambda_1' = k'^n\left\{\sin\operatorname{coam}\left(\frac{2K'}{n},k'\right)\sin\operatorname{coam}\left(\frac{4K'}{n},k'\right)\cdots\sin\operatorname{coam}\left(\frac{(n-1)K'}{n},k'\right)\right\}^4$$

$$M_1 = \left\{\frac{\sin\operatorname{coam}\left(\frac{2K'}{n},k'\right)\sin\operatorname{coam}\left(\frac{4K'}{n},k'\right)\sin\operatorname{coam}\cdots\left(\frac{(n-1)K'}{n},k'\right)}{\sin\operatorname{am}\left(\frac{2K'}{n},k'\right)\sin\operatorname{am}\left(\frac{4K'}{n},k'\right)\sin\operatorname{am}\cdots\left(\frac{(n-1)K'}{n},k'\right)}\right\}^2$$

$$\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) = \frac{\frac{\sin\operatorname{am}u}{M_1}\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{2K'}{n},k'\right)}\right\}\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{4K'}{n},k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{(n-1)K'}{n},k'\right)}\right\}}{\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{K'}{n},k'\right)}\right\}\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{3K'}{n},k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{(n-2)K'}{n},k'\right)}\right\}}$$

$$\cos\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) = \frac{\cos\operatorname{am}u\left\{1-\sin^2\operatorname{am}u\Delta^2\operatorname{am}\left(\frac{2K'}{n}\right)\right\}\left\{1-\sin^2\operatorname{am}u\Delta^2\operatorname{am}\left(\frac{4K'}{n}\right)\right\}\left\{1-\sin^2\operatorname{am}u\Delta^2\operatorname{am}\left(\frac{(n-1)K'}{n}\right)\right\}}{\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{K'}{n},k'\right)}\right\}\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{3K'}{n},k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{(n-2)K'}{n},k'\right)}\right\}}$$

$$\Delta\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) = \frac{\Delta\operatorname{am}u\left\{1-\sin^2\operatorname{am}u\Delta^2\operatorname{am}\left(\frac{K'}{n}\right)\right\}\left\{1-\sin^2\operatorname{am}u\Delta^2\operatorname{am}\left(\frac{3K'}{n}\right)\right\}\left\{1-\sin^2\operatorname{am}u\Delta^2\operatorname{am}\left(\frac{(n-2)K'}{n}\right)\right\}}{\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{K'}{n},k'\right)}\right\}\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{3K'}{n},k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\operatorname{am}u}{\tan^2\operatorname{am}\left(\frac{(n-2)K'}{n},k'\right)}\right\}}$$

$$\sqrt{\frac{1-\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)}{1+\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)}}$$

$$= \sqrt{\frac{1-\sin\operatorname{am}u}{1+\sin\operatorname{am}u}}\cdot\frac{\left\{1-\sin\operatorname{am}u\Delta\operatorname{am}\left(\frac{2K'}{n},k'\right)\right\}\left\{1-\sin\operatorname{am}u\Delta\operatorname{am}\left(\frac{4K'}{n},k'\right)\right\}\cdots\left\{1-\sin\operatorname{am}u\Delta\operatorname{am}\left(\frac{(n-1)K'}{n},k'\right)\right\}}{\left\{1+\sin\operatorname{am}u\Delta\operatorname{am}\left(\frac{2K'}{n},k'\right)\right\}\left\{1+\sin\operatorname{am}u\Delta\operatorname{am}\left(\frac{4K'}{n},k'\right)\right\}\cdots\left\{1+\sin\operatorname{am}u\Delta\operatorname{am}\left(\frac{(n-1)K'}{n},k'\right)\right\}}$$

$$\sqrt{\frac{1-\lambda_1\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)}{1+\lambda_1\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)}}$$

$$= \sqrt{\frac{1-k\sin\operatorname{am}u}{1+k\sin\operatorname{am}u}}\cdot\frac{\left\{1-\Delta\operatorname{am}\left(\frac{K'}{n},k'\right)\sin\operatorname{am}u\right\}\left\{1-\Delta\operatorname{am}\left(\frac{3K'}{n},k'\right)\sin\operatorname{am}u\right\}\cdots\left\{1-\Delta\operatorname{am}\left(\frac{(n-2)K'}{n},k'\right)\sin\operatorname{am}u\right\}}{\left\{1+\Delta\operatorname{am}\left(\frac{K'}{n},k'\right)\sin\operatorname{am}u\right\}\left\{1+\Delta\operatorname{am}\left(\frac{3K'}{n},k'\right)\sin\operatorname{am}u\right\}\cdots\left\{1+\Delta\operatorname{am}\left(\frac{(n-2)K'}{n},k'\right)\sin\operatorname{am}u\right\}}$$

$$\frac{\lambda_1}{kM_1}\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) = \sin\operatorname{am}u+\frac{2}{k}\sum\frac{\Delta\operatorname{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin\operatorname{am}u}{\sin^2\operatorname{am}\left(\frac{(2q-1)K'}{n},k'\right)+\cos^2\operatorname{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin^2\operatorname{am}u}$$

$$\frac{(-1)^{\frac{n-1}{2}}\lambda_1}{kM_1}\cos\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) = \cos\operatorname{am}u-\frac{2(-1)^{\frac{n-1}{2}}}{k}\sum\frac{(-1)^q\sin\operatorname{am}\left(\frac{(2q-1)K'}{n},k'\right)\Delta\operatorname{am}\left(\frac{(2q-1)K'}{n},k'\right)\cos\operatorname{am}u}{\sin^2\operatorname{am}\left(\frac{(2q-1)K'}{n},k'\right)+\cos^2\operatorname{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin^2\operatorname{am}u}$$

$$\frac{(-1)^{\frac{n-1}{2}}}{M_1}\Delta\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) = \Delta\operatorname{am}u\quad-2(-1)^{\frac{n-1}{2}}\sum\frac{(-1)^q\sin\operatorname{am}\frac{(2q-1)iK'}{n}\Delta\operatorname{am}u}{\sin^2\operatorname{am}\left(\frac{(2q-1)K'}{n},k'\right)+\cos^2\operatorname{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin^2\operatorname{am}u}$$

$$\frac{\lambda_1'}{k'M_1}\tan\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right) = \tan\operatorname{am}u+2\sum\frac{(-1)^q\cos\operatorname{am}\left(\frac{2qK'}{n},k'\right)\Delta\operatorname{am}\left(\frac{2qK'}{n},k'\right)\sin\operatorname{am}u\cos\operatorname{am}u}{1-\Delta^2\operatorname{am}\left(\frac{2qK'}{n},k'\right)\sin^2\operatorname{am}u}.$$

In the formulas for the first transformation $(-1)^{\frac{n-1}{2}}M$ was put instead of M. It was convenient to exhibit the formulas for the second transformation in two ways, both in imaginary and real form, in which additionally $\frac{1}{\sin\operatorname{am}\frac{(n-2m)miK'}{n}}$, $\frac{1}{\cos\operatorname{am}\frac{(n-2m)miK'}{n}}$ etc. was written instead of $k\sin\operatorname{am}\frac{2miK'}{n}$, $k\cos\operatorname{am}\frac{2miK'}{n}$ etc. everywhere: this was, as the reduction to the real form, easily done by means of the formulas given in § 19. Where the ambiguous sign $\pm$ was appears, the first $+$ is to be chosen, if $\frac{n-1}{2}$ is an even number, the other $-$, if $\frac{n-1}{2}$ is an odd number; the contrary holds for the sign $\mp$. In the sums denoted by the prefixed sign $\sum$ the values $1, 2, 3, \cdots, \frac{n-1}{2}$ are to be assigned to the number q.

From the formulas propounded for the first transformation it is clear, if u successively takes on the values:

$$0,\quad \frac{K}{n},\quad \frac{2K}{n},\quad \frac{3K}{n},\quad \frac{4K}{n},\cdots,$$

that $\operatorname{am}\left(\frac{u}{M},\lambda\right)$ will be:

$$0,\quad \frac{\pi}{2},\quad \pi,\quad \frac{3\pi}{2},\quad 2\pi,\cdots,$$

whence we obtain:

$$\frac{K}{nM}=\Lambda.$$

On the other hand, we have seen in the second transformation, if u is: 0, K, $2K$, $3K,\cdots$ or $\operatorname{am}u$: 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2},\cdots$, that also $\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)$ is: 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2},\cdots$, whence in this case it is:

$$\frac{K}{M_1} = \Lambda_1.$$

By the way, it is obvious from the formulas for the moduli λ, λ', λ_1, λ_1' while n increases that the moduli λ, λ_1' converge to zero rapidly, and hence at the same time the moduli λ', λ_1 get very close to 1. Therefore, it is convenient to call the first transformation of the modulus *transformation of the greater into the smaller*, the second *transformation of the smaller into the greater*.

1.14 On complementary Transformations or how from the Transformation of one Modulus into another the Transformation of one Complement into another is derived

25.

In the formulas found above:

$$\tan\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{k'^n}{\lambda'}}\tan\operatorname{am} u\tan\operatorname{am}(u+4\omega)\tan\operatorname{am}(u+8\omega)\cdots\tan\operatorname{am}(u+4(n-1)\omega)$$

let us put $u = iu'$, $\omega = i\omega'$ so that it is $\omega = \frac{mK+m'iK'}{n}$, $\omega' = \frac{m'K'-miK}{n}$. But it is (§ 19):

$$\begin{aligned}\tan\operatorname{am}(iu',k) &= i\sin\operatorname{am}(u',k')\\ \tan\operatorname{am}(iu',\lambda) &= i\sin\operatorname{am}(u',\lambda'),\end{aligned}$$

whence we see the mentioned formula to goes over into the following:

$$\sin\operatorname{am}\left(\frac{u'}{M},\lambda'\right) = (-1)^{\frac{n-1}{2}}\sin\operatorname{am} u'\sin\operatorname{am}(u'+4\omega')\sin\operatorname{am}(u'+8\omega')\cdots\sin\operatorname{am}(u'+4(n-1)\omega')\ (\text{mod. } k')$$

Further, we found:

$$\begin{aligned}\lambda' &= \frac{k'^n}{[\Delta\operatorname{am}2\omega\Delta\operatorname{am}4\omega\cdots\Delta\operatorname{am}4(n-1)\omega]^4}\\ M &= (-1)^{\frac{n-1}{2}}\frac{[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}(n-1)\omega]^2}{[\sin\operatorname{am}2\omega\sin\operatorname{am}4\omega\cdots\sin\operatorname{am}(n-1)\omega]^2}\end{aligned}$$

which from the formulas:

$$\begin{aligned}\Delta \operatorname{am}(iu,k) &= \frac{1}{\sin\operatorname{coam}(u,k')}\\ \sin\operatorname{coam}(iu,k) &= \frac{1}{\Delta\operatorname{am}(u,k')},\end{aligned}$$

whence it also follows:

$$\frac{\sin\operatorname{coam}(iu,k)}{\sin\operatorname{am}(iu,k)} = \frac{-i}{\tan\operatorname{am}(u,k')\Delta\operatorname{am}(u,k')} = \frac{-i\sin\operatorname{coam}(u,k')}{\sin\operatorname{am}(u,k')}$$

go over into the following:

$$\begin{aligned}\lambda' &= k'^n[\sin\operatorname{coam}2\omega'\sin\operatorname{coam}4\omega'\sin\operatorname{coam}(n-1)\omega']^4 \quad (\text{mod. } k')\\ M &= \frac{[\sin\operatorname{coam}2\omega'\sin\operatorname{coam}4\omega'\sin\operatorname{coam}(n-1)\omega']^2}{[\sin\operatorname{am}2\omega'\sin\operatorname{am}4\omega'\sin\operatorname{am}(n-1)\omega']^2} \quad (\text{mod. } k')\end{aligned}$$

Having compared these formulas to those which serve for the transformation of the modulus k into the modulus λ:

$$\begin{aligned}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) &= \sqrt{\frac{k^n}{\lambda}}\sin\operatorname{am}u\sin\operatorname{am}(u+4\omega)\sin\operatorname{am}(u+8\omega)\cdots\sin\operatorname{am}(u+4(n-1)\omega)\\ \lambda &= k^n[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}(n-1)\omega]^4\\ M &= (-1)^{\frac{n-1}{2}}\frac{[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}(n-1)\omega]^2}{[\sin\operatorname{am}2\omega\sin\operatorname{am}4\omega\cdots\sin\operatorname{am}(n-1)\omega]^2}\end{aligned}$$

this reveals a theorem, which has to be considered to be of highest importance in the theory of transformations:

Whatever formulas on the transformation of the modulus k into the modulus λ can be propounded, the same hold, having changed k into k', λ into λ', ω into $\omega' = \frac{\omega}{i}$, M into $(-1)^{\frac{n-1}{2}}M$.

But we will call the transformation of the complement into another complement, derived from the propounded transformation in the way just described, *complementary transformation.*

It is easily seen that the real transformations of the modulus k' are the complementary ones of the real transformation of the modulus k, such that nevertheless the second of the modulus k' is the complementary of the first of the modulus k, and the first of the modulus k' is the complementary of the second of the modulus k. For, if in the theorem just propounded one puts $\omega = \frac{\pm K}{n}$, $\omega = \frac{\pm iK'}{n}$ which corresponds to the first and second transformations of the modulus k, it is $\omega' = \frac{\omega}{i} = \frac{\mp iK}{n}$, $\omega' = \frac{\omega}{i} = \frac{\pm K'}{n}$, which corresponds to the second and first transformations of the modulus k', respectively. Because while the modulus grows the complement decreases and vice versa, if the transformation of the complement into the complement is the transformation of the greater into the smaller, the transformation of the complement or the complementary transformation must be the one of the smaller into the greater and vice versa. Therefore, we see, having changed k into k', that λ goes over into λ_1', λ_1 goes over into λ'. Only the multiplicator M, common to the first transformation and its complementary counterpart, will go over into M_1, which belongs to the second transformation and its complementary counterpart, and vice versa M_1 into M. Hence from the formulas found above:

$$\Lambda = \frac{K}{nM}, \quad \Lambda_1 = \frac{K}{M_1}$$

these ones follow:

$$\Lambda_1' = \frac{K'}{nM_1'}, \quad \Lambda' = \frac{K'}{M'},$$

whence these formulas of highest importance in this theory emerge:

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}; \quad \frac{\Lambda_1'}{\Lambda_1} = \frac{1}{n} \cdot \frac{K'}{K}.$$

These formulas define the genuine character of the propounded transformation, whence it is clear that we justly referred the particular transformations to the particular numbers n. I mention, if n was a composite number $= n'n''$, that from the particular real roots of the modular equations or from the particular real moduli into which the given modulus k can be transformed by means of a substitution of n-th order one reaches equations of this kind:

$$\frac{\Lambda'}{\Lambda} = \frac{n'}{n} \cdot \frac{K'}{K},$$

which correspond to the particular factorization of the number n into two factors. Therefore, from their total number, if n was a square, it will also be:

$$\frac{\Lambda'}{\Lambda} = \frac{K'}{K}, \quad \text{whence} \quad \lambda = k,$$

which in the case in which n is a square tells us that from the total number of substitutions there is one which yields a multiplication.

1.15 On supplementary Transformations for Multiplication

26.

Let us recall the formulas:

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}, \quad \frac{\Lambda'_1}{\Lambda_1} = \frac{1}{n} \cdot \frac{K'}{K},$$

having written which in this way:

$$\begin{aligned} \frac{\Lambda'}{\Lambda} &= n\frac{K'}{K} \\ \frac{K'}{K} &= n\frac{\Lambda'_1}{\Lambda_1}, \end{aligned}$$

it becomes clear *that the modulus λ depends on the modulus k in the same way as the modulus k depends on the modulus λ_1, or the the modulus k depends on the modulus λ in the same way as the modulus λ_1 depends on the modulus k.* Therefore, by means of the first transformation or of the greater into the smaller, in which k is transformed into λ, λ_1 will be transformed into k; by means of the second transformation or of the smaller into the greater, in which k is transformed into λ_1, λ will be transformed into k. Therefore, *after the first transformation after having used the second before or after the second after having used the first before the modulus k is transformed into itself, or the first and the second transformation applied successively, in an arbitrary order, yield a multiplication*.

Let us denote the multiplicator which depends on λ in the same way as M_1 depends on k by M', and let us denote the multiplicator which depends on λ_1 in the same way as M depends on k by M_1 such that the following equations are obtained:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$
$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{dy}{M'\sqrt{(1-y^2)(1-k^2y^2)}},$$

of which the one corresponds to the transformation of the modulus k into the modulus λ by means of the first transformation, the other to the transformation of the modulus λ into the modulus k by means of the second transformation. From these equations it results:

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{dx}{MM'\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \text{whence} \quad z = \sin\operatorname{am}\left(\frac{u}{MM'}\right).$$

But from the equation $\Lambda_1 = \frac{K}{M_1}$ by changing k into λ, having done which K goes over into Λ, λ_1 into k, Λ_1 into K, M_1 into M', one obtains $K = \frac{\Lambda}{M'}$ having compared which equation to $\Lambda = \frac{K}{nM}$ $\frac{1}{MM'} = n$ results, whence it is:

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{ndx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

In the same way, from the equation $\Lambda = \frac{K}{nM}$ by changing k into λ_1, having done which K goes over into Λ_1, λ into k, Λ into K, M_1 into M_1', $K = \frac{\Lambda_1}{nM_1'}$, after having compared which equation to $\Lambda_1 = \frac{K}{M_1}$, this yields $\frac{1}{M_1M_1'} = n$; hence we see that in those two cases after two successively applied transformations the argument is multiplied by the number n.

If after the transformation of the modulus k into the modulus λ λ is then again transformed back into the modulus k such that a multiplication results we will call this transformation *the supplementary transformation for multiplication* of the latter or simply *supplementary*.

Let us both for the sake of an example and the use for the following list the formulas for the *supplementary* transformation of the *first* or of the modulus λ into the modulus k which transformation will be the second transformation of λ; we will nevertheless only list them in the imaginary form, since the reduction to the real one is easily done. We immediately obtain these formulas, if in those which were propounded above for the second transformation of the modulus k (confer Table II, A. §24) we put λ instead of k, k instead of λ_1, $\frac{u}{M}$ instead of u, $M = \frac{1}{nM}$ instead of M_1, whence $\frac{u}{MM'} = nu$ instead of $\frac{u}{M_1}$. In these formulas, but only in those, the modulus will be λ, if the modulus k is not explicitly added; furthermore, for the sake of brevity we put $y = \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)$; as above one has to assign the following values to the number q:

$$1, 2, 3, \cdots, \frac{n-1}{2}.$$

1.16 Formulas for the Transformation of the Modulus λ into the Modulus k or the supplementary of the first

27.

$$\begin{aligned}
k &= \lambda^n \left\{ \sin\operatorname{coam}\frac{2i\Lambda'}{n} \sin\operatorname{coam}\frac{4i\Lambda'}{n} \cdots \sin\operatorname{coam}\frac{(n-1)i\Lambda'}{n} \right\}^4 \\
k' &= \frac{\lambda'^n}{\left\{ \Delta\operatorname{am}\frac{2i\Lambda'}{n} \Delta\operatorname{am}\frac{4i\Lambda'}{n} \cdots \Delta\operatorname{am}\frac{(n-1)i\Lambda'}{n} \right\}^4} \\
\frac{1}{nM} &= (-1)^{\frac{n-1}{2}} \left\{ \frac{\sin\operatorname{coam}\frac{2i\Lambda'}{n} \sin\operatorname{coam}\frac{4i\Lambda'}{n} \cdots \sin\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}{\sin\operatorname{am}\frac{2i\Lambda'}{n} \sin\operatorname{am}\frac{4i\Lambda'}{n} \cdots \sin\operatorname{am}\frac{(n-1)i\Lambda'}{n}} \right\}^2
\end{aligned}$$

$$
\begin{aligned}
\sin\operatorname{am}(nu,k) &= \frac{nMy\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{2i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{4i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{(n-1)i\Lambda'}{n}}\right)}{\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{3i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{(n-2)i\Lambda'}{n}}\right)} \\
&= \sqrt{\frac{\lambda^n}{k}}\sin\operatorname{am}\frac{u}{M}\sin\operatorname{am}\left(\frac{u}{M}+\frac{4i\Lambda'}{n}\right)\sin\operatorname{am}\left(\frac{u}{M}+\frac{8i\Lambda'}{n}\right)\cdots\sin\operatorname{am}\left(\frac{u}{M}+\frac{4(n-1)i\Lambda'}{n}\right) \\
\cos\operatorname{am}(nu,k) &= \frac{\sqrt{1-y^2}\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{2i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{4i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}\right)}{\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{3i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{(n-2)i\Lambda'}{n}}\right)} \\
&= \sqrt{\frac{k'\lambda^{n\,n}}{k\lambda'}}\cos\operatorname{am}\frac{u}{M}\cos\operatorname{am}\left(\frac{u}{M}+\frac{4i\Lambda'}{n}\right)\cos\operatorname{am}\left(\frac{u}{M}+\frac{8i\Lambda'}{n}\right)\cdots\cos\operatorname{am}\left(\frac{u}{M}+\frac{4(n-1)i\Lambda'}{n}\right) \\
\Delta\operatorname{am}(nu,k) &= \frac{\sqrt{1-\lambda^2y^2}\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{3i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-2)i\Lambda'}{n}}\right)}{\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{3i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{(n-2)i\Lambda'}{n}}\right)} \\
&= \sqrt{\frac{k'}{\lambda'^n}}\Delta\operatorname{am}\frac{u}{M}\cos\operatorname{am}\left(\frac{u}{M}+\frac{4i\Lambda'}{n}\right)\Delta\operatorname{am}\left(\frac{u}{M}+\frac{8i\Lambda'}{n}\right)\cdots\Delta\operatorname{am}\left(\frac{u}{M}+\frac{4(n-1)i\Lambda'}{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{\frac{1-\sin\operatorname{am}(nu,k)}{1+\sin\operatorname{am}(nu,k)}} &= \sqrt{\frac{1-y}{1+y}}\cdot\frac{\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{2i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{4i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}\right)}{\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{2i\Lambda'}{n}}\right)\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{4i\Lambda'}{n}}\right)\cdots\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}\right)} \\
\sqrt{\frac{1-k\sin\operatorname{am}(nu,k)}{1+k\sin\operatorname{am}(nu,k)}} &= \sqrt{\frac{1-\lambda y}{1+\lambda y}}\cdot\frac{\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{i\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{3i\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-2)i\Lambda'}{n}}\right)}{\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{i\Lambda'}{n}}\right)\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{3i\Lambda'}{n}}\right)\cdots\left(1+\frac{y^2}{\sin^2\operatorname{coam}\frac{(n-2)i\Lambda'}{n}}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\sin\operatorname{am}(nu,k) &= \frac{\lambda y}{knM}-\frac{2y}{knM}\sum\frac{\cos\operatorname{am}\frac{(2q-1)i\Lambda'}{n}\Delta\operatorname{am}\frac{(2q-1)i\Lambda'}{n}}{\sin^2\operatorname{am}\frac{(2q-1)i\Lambda'}{n}-y^2} \\
\cos\operatorname{am}(nu,k) &= \frac{(-1)^{\frac{n-1}{2}}\lambda\sqrt{1-y^2}}{knM}+\frac{2\sqrt{1-y^2}}{iknM}\sum\frac{(-1)^q\sin\operatorname{am}\frac{(2q-1)i\Lambda'}{n}\Delta\operatorname{am}\frac{(2q-1)i\Lambda'}{n}}{\sin^2\operatorname{am}\frac{(2q-1)i\Lambda'}{n}-y^2} \\
\Delta\operatorname{am}(nu,k) &= \frac{(-1)^{\frac{n-1}{2}}}{nM}\sqrt{1-\lambda^2y^2}+\frac{2\sqrt{1-\lambda^2y^2}}{inM}\sum\frac{(-1)^q\sin\operatorname{am}\frac{(2q-1)i\Lambda'}{n}\cos\operatorname{am}\frac{(2q-1)i\Lambda'}{n}}{\sin^2\operatorname{am}\frac{(2q-1)i\Lambda'}{n}-y^2} \\
\tan\operatorname{am}(nu,k) &= \frac{\lambda'}{k'nM}\cdot\frac{y}{\sqrt{1-y^2}}+\frac{2\lambda'y\sqrt{1-y^2}}{k'nM}\sum\frac{(-1)^q\Delta\operatorname{am}\frac{2qi\Lambda'}{n}}{\cos^2\operatorname{am}\frac{2qi\Lambda'}{n}-y^2\Delta^2\operatorname{am}\frac{2qi\Lambda'}{n}}.
\end{aligned}
$$

I already communicated this general analytical theorem on the supplementary transformation of the first to Legendre at the beginning of August 1827, which he also wanted to mention in the note mentioned above (*Nova Astronomica* a. 1827, Nr. 130). A similar system of formulas for the other supplementary transformation of the second or the transformation of the modulus λ into the modulus k could have been stated. To render all these things more clear, it was convenient to give a complete list of the fundamental formulas for the first and second transformation and their complementary and supplementary transformation in the added table.

Only one of the total number of imaginary transformations has a supplementary one for multiplication. Let us suppose, which is possible, that the numbers m, m' in § 20 do not have a common factor: Further, let $m\mu' - \mu m' = 1$ μ, μ' denoting whole positive or negative integers. Now, if one puts $\omega = \frac{\nu K + \mu' iK'}{nM}$ in our general formulas propounded on the transformation in § 20 and k and λ are interchanged, one obtains formulas extending to the supplementary of the transformation. Having put $m = 1$, $m' = 0$ it is $\mu = 0$, $\mu' = 1$, whence $\frac{\mu K + \mu' iK'}{nM} = \frac{iK'}{nM} = \frac{i\Lambda'}{n}$ which yields the supplementary of the first, as we saw already.

A. First Transformation with Supplementary

$$(a) \qquad \lambda = k^n \sin^4 \operatorname{coam} \frac{2K}{n} \sin^4 \operatorname{coam} \frac{4K}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)K}{n} \qquad (\text{mod. } k)$$

$$(aa) \qquad k^n = \lambda^n \sin^4 \operatorname{coam} \frac{2i\Lambda'}{n} \sin^4 \operatorname{coam} \frac{4i\Lambda'}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)i\Lambda'}{n} \qquad (\text{mod. } \lambda)$$

$$= \frac{\lambda^n}{\Delta^4 \operatorname{am} \frac{2\Lambda'}{n} \Delta^4 \operatorname{am} \frac{4\Lambda'}{n} \cdots \Delta^4 \operatorname{am} \frac{(n-1)\Lambda'}{n}} \qquad (\text{mod. } \lambda')$$

$$(b) \qquad M = \frac{\sin^2 \operatorname{coam} \frac{2K}{n} \sin^2 \operatorname{coam} \frac{4K}{n} \cdots \sin^2 \operatorname{coam} \frac{(n-1)K}{n}}{\sin^2 \operatorname{am} \frac{2K}{n} \sin^2 \operatorname{am} \frac{4K}{n} \cdots \sin^2 \operatorname{am} \frac{(n-1)K}{n}} \qquad (\text{mod. } (k)$$

$$(bb) \qquad \frac{1}{nM} = \frac{\sin^2 \operatorname{coam} \frac{2\Lambda'}{n} \sin^2 \operatorname{coam} \frac{4\Lambda'}{n} \cdots \sin^2 \operatorname{coam} \frac{(n-1)\Lambda'}{n}}{\sin^2 \operatorname{am} \frac{2\Lambda'}{n} \sin^2 \operatorname{am} \frac{4\Lambda'}{n} \cdots \sin^2 \operatorname{am} \frac{(n-1)\Lambda'}{n}} \qquad (\text{mod. } \lambda')$$

$$\sin \operatorname{am}(u, k) = x; \quad \sin \operatorname{am}\left(\frac{u}{M}, \lambda\right) = y; \quad \sin \operatorname{am}(nu, k) = z$$

$$(c)\qquad y = (-1)^{\frac{n-1}{2}}\sqrt{\frac{k^n}{\lambda}}\sin\operatorname{am} u \sin\operatorname{am}\left(u+\frac{4K}{n}\right)\sin\operatorname{am}\left(u+\frac{8K}{n}\right)\sin\operatorname{am}\cdots\left(u+\frac{4(n-1)K}{n}\right)\qquad (\text{mod. } k)$$

$$= \frac{\frac{x}{M}\left(1-\frac{x^2}{\sin^2\operatorname{am}\frac{2K}{n}}\right)\left(1-\frac{x^2}{\sin^2\operatorname{am}\frac{4K}{n}}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{am}\frac{(n-1)K}{n}}\right)}{\left(1-k^2x^2\sin^2\operatorname{am}\frac{2K}{n}\right)\left(1-k^2x^2\sin^2\operatorname{am}\frac{4K}{n}\right)\cdots\left(1-k^2x^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\right)}\qquad (\text{mod. } k)$$

$$(cc)\qquad z = \sqrt{\frac{\lambda^n}{k}}\sin\operatorname{am}\frac{u}{M}\sin\operatorname{am}\left(\frac{u}{M}+\frac{4i\Lambda'}{n}\right)\sin\operatorname{am}\left(\frac{u}{M}+\frac{8i\Lambda'}{n}\right)\cdots\sin\operatorname{am}\left(\frac{u}{M}+\frac{4(n-1)i\Lambda'}{n}\right)\qquad (\text{mod. } \lambda)$$

$$= \frac{nMy\left(1+\frac{y^2}{\tan^2\operatorname{am}\frac{2\Lambda'}{n}}\right)\left(1+\frac{y^2}{\tan^2\operatorname{am}\frac{4\Lambda'}{n}}\right)\cdots\left(1+\frac{y^2}{\tan^2\operatorname{am}\frac{(n-1)\Lambda'}{n}}\right)}{\left(1+\lambda^2y^2\tan^2\operatorname{am}\frac{2\Lambda'}{n}\right)\left(1+\lambda^2y^2\tan^2\operatorname{am}\frac{4\Lambda'}{n}\right)\cdots\left(1+\lambda^2y^2\tan^2\operatorname{am}\frac{(n-1)\Lambda'}{n}\right)}\qquad (\text{mod. } \lambda')$$

Complementary Transformations

$$(a)\qquad \lambda' = k'^n\sin^4\operatorname{coam}\frac{2iK}{n}\sin^4\operatorname{coam}\frac{4iK}{n}\cdots\sin^4\operatorname{coam}\frac{(n-1)iK}{n}\qquad (\text{mod. } k')$$

$$= \frac{k'^n}{\Delta^4\operatorname{am}\frac{2K}{n}\Delta^4\operatorname{am}\frac{4K}{n}\cdots\Delta^4\operatorname{am}\frac{(n-1)K}{n}}\qquad (\text{mod. } k)$$

$$(aa)\qquad k' = \lambda'^n\sin^4\operatorname{coam}\frac{2\Lambda'}{n}\sin^4\operatorname{coam}\frac{4\Lambda'}{n}\cdots\sin^4\operatorname{coam}\frac{(n-1)\Lambda'}{n}\qquad (\text{mod. } \lambda')$$

(b) and (bb) are the same as above.

$$\sin\operatorname{am}(u,k') = x;\quad \sin\operatorname{am}\left(\frac{u}{M},\lambda'\right) = y;\quad \sin\operatorname{am}(nu,k') = z$$

$$(c)\qquad y = \sqrt{\frac{k'^n}{\lambda'}}\sin\operatorname{am} u\sin\operatorname{am}\left(u+\frac{4iK}{n}\right)\sin\operatorname{am}\left(u+\frac{8iK}{n}\right)\cdots\sin\operatorname{am}\left(u+\frac{4(n-1)iK}{n}\right)\qquad (\text{mod. } k')$$

$$= \frac{\frac{x}{M}\left(1+\frac{x^2}{\tan^2\operatorname{am}\frac{2K}{n}}\right)\left(1+\frac{x^2}{\tan^2\operatorname{am}\frac{4K}{n}}\right)\cdots\left(1+\frac{x^2}{\tan^2\operatorname{am}\frac{(n-1)K}{n}}\right)}{\left(1+k'^2x^2\tan^2\operatorname{am}\frac{2K}{n}\right)\left(1+k'^2x^2\tan^2\operatorname{am}\frac{4K}{n}\right)\cdots\left(1+k'^2x^2\tan^2\operatorname{am}\frac{(n-1)K}{n}\right)}\qquad (\text{mod. } k)$$

$$(cc)\qquad z = (-1)^{\frac{n-1}{2}}\sin\operatorname{am}\frac{u}{M}\sin\operatorname{am}\left(\frac{u}{M}+\frac{4\Lambda'}{n}\right)\left(\frac{u}{M}+\frac{8\Lambda'}{n}\right)\cdots\left(\frac{u}{M}+\frac{4(n-1)\Lambda'}{n}\right)\qquad (\text{mod. } \lambda')$$

$$= \frac{nMy\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{2\Lambda'}{n}}\right)\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{4\Lambda'}{n}}\right)\cdots\left(1-\frac{y^2}{\sin^2\operatorname{am}\frac{(n-1)\Lambda'}{n}}\right)}{\left(1-\lambda'^2y^2\sin^2\operatorname{am}\frac{2\Lambda'}{n}\right)\left(1-\lambda'^2y^2\sin^2\operatorname{am}\frac{4\Lambda'}{n}\right)\cdots\left(1-\lambda'^2y^2\sin^2\operatorname{am}\frac{(n-1)\Lambda'}{n}\right)}\qquad (\text{mod. } \lambda')$$

$$\Lambda = \frac{K}{nM};\quad \Lambda' = \frac{K'}{M}$$

B. Second Transformation with Supplementary

$$
\begin{aligned}
&(a) && \lambda_1 && = k^n \sin^4 \operatorname{coam} \frac{2iK'}{n} \sin^4 \operatorname{coam} \frac{4iK'}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)iK'}{n} && (\text{mod. } k) \\
& && && = \frac{k^n}{\Delta^4 \operatorname{am} \frac{2K'}{n} \Delta^4 \operatorname{am} \frac{4K'}{n} \cdots \Delta^4 \operatorname{am} \frac{(n-1)K'}{n}} && (\text{mod. } k') \\
&(aa) && k && = \lambda_1^n \sin^4 \operatorname{coam} \frac{2\Lambda_1}{n} \sin^4 \operatorname{coam} \frac{2\Lambda_1}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)\Lambda_1}{n} && (\text{mod. } \lambda_1) \\
&(b) && M_1 && = \frac{\sin^2 \operatorname{coam} \frac{2K'}{n} \sin^2 \operatorname{coam} \frac{4K'}{n} \cdots \sin^2 \operatorname{coam} \frac{(n-1)K'}{n}}{\sin^2 \operatorname{am} \frac{2K'}{n} \sin^2 \operatorname{am} \frac{4K'}{n} \cdots \sin^2 \operatorname{am} \frac{(n-1)K'}{n}} && (\text{mod. } k') \\
&(bb) && \frac{1}{nM_1} && = \frac{\sin^2 \operatorname{coam} \frac{2\Lambda_1}{n} \sin^2 \operatorname{coam} \frac{4\Lambda_1}{n} \cdots \sin^2 \operatorname{coam} \frac{(n-1)\Lambda_1}{n}}{\sin^2 \operatorname{am} \frac{2\Lambda_1}{n} \sin^2 \operatorname{am} \frac{4\Lambda_1}{n} \cdots \sin^2 \operatorname{am} \frac{(n-1)\Lambda_1}{n}} && (\text{mod. } \lambda_1)
\end{aligned}
$$

$$
\sin \operatorname{am}(u, k) = x; \quad \sin \operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right) = y; \quad \sin \operatorname{am}(nu, k) = z
$$

$$
\begin{aligned}
&(c) \quad y = \sqrt{\frac{k^n}{\lambda_1}} \sin \operatorname{am} u \sin \operatorname{am}\left(u + \frac{4iK'}{n}\right) \sin \operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots \sin \operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right) && (\text{mod. } k) \\
&\quad = \frac{\frac{x}{M_1}\left(1 + \frac{x^2}{\tan^2 \operatorname{am} \frac{2K'}{n}}\right)\left(1 + \frac{x^2}{\tan^2 \operatorname{am} \frac{4K'}{n}}\right) \cdots \left(1 + \frac{x^2}{\tan^2 \operatorname{am} \frac{(n-1)K'}{n}}\right)}{\left(1 + k^2x^2 \tan^2 \operatorname{am} \frac{2K'}{n}\right)\left(1 + k^2x^2 \tan^2 \operatorname{am} \frac{4K'}{n}\right) \cdots \left(1 + k^2x^2 \tan^2 \operatorname{am} \frac{(n-1)K'}{n}\right)} && (\text{mod. } k') \\
&(cc) \quad z = (-1)^{\frac{n-1}{2}} \sqrt{\frac{\lambda_1^n}{k}} \sin \operatorname{am} \frac{u}{M_1} \sin \operatorname{am}\left(\frac{u}{M_1} + \frac{4\Lambda_1}{n}\right) \sin \operatorname{am}\left(\frac{u}{M_1} + \frac{8\Lambda_1}{n}\right) \cdots \sin \operatorname{am}\left(\frac{u}{M_1} + \frac{4(n-1)\Lambda_1}{n}\right) && (\text{mod. } \lambda_1) \\
&\quad = \frac{nM_1 y \left(1 - \frac{y^2}{\sin^2 \operatorname{am} \frac{2\Lambda_1}{n}}\right)\left(1 - \frac{y^2}{\sin^2 \operatorname{am} \frac{4\Lambda_1}{n}}\right) \cdots \left(1 - \frac{y^2}{\sin^2 \operatorname{am} \frac{(n-1)\Lambda_1}{n}}\right)}{\left(1 - \lambda_1^2 y^2 \sin^2 \operatorname{am} \frac{2\Lambda_1}{n}\right)\left(1 - \lambda_1^2 y^2 \sin^2 \operatorname{am} \frac{4\Lambda_1}{n}\right) \cdots \left(1 - \lambda_1^2 y^2 \sin^2 \operatorname{am} \frac{(n-1)\Lambda_1}{n}\right)} && (\text{mod. } \lambda_1)
\end{aligned}
$$

Complementary Transformations

$$
\begin{aligned}
&(a) && \lambda_1' = k'^n \sin^4 \operatorname{coam} \frac{2K'}{n} \sin^4 \operatorname{coam} \frac{4K'}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)K'}{n} && (\text{mod. } k') \\
&(aa) && k' = \lambda_1'^n \sin^4 \operatorname{coam} \frac{2i\Lambda_1}{n} \sin^4 \operatorname{coam} \frac{4i\Lambda_1}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)i\Lambda_1}{n} && (\text{mod. } \lambda_1') \\
& && \quad = \frac{\lambda_1'^n}{\Delta^4 \operatorname{am} \frac{2\Lambda_1}{n} \Delta^4 \operatorname{am} \frac{2\Lambda_1}{n} \cdots \Delta^4 \operatorname{am} \frac{2\Lambda_1}{n}} && (\text{mod. } \lambda_1)
\end{aligned}
$$

(b) and (bb) are the same as above.

$$
\sin \operatorname{am}(u, k') = x; \quad \sin \operatorname{am}\left(\frac{u}{M_1}, \lambda_1'\right) = y; \quad \sin \operatorname{am}(nu, k') = z
$$

$$(c) \quad y = (-1)^{\frac{n-1}{2}} \sqrt{\frac{k'^n}{\lambda_1'}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4K'}{n}\right) \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{8K'}{n}\right) \cdots \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4(n-1)K'}{n}\right) \quad (\text{mod. } k')$$

$$= \frac{\frac{x}{M_1}\left(1 - \frac{x^2}{\sin^2\operatorname{am}\frac{2K'}{n}}\right)\left(1 - \frac{x^2}{\sin^2\operatorname{am}\frac{4K'}{n}}\right)\cdots\left(1 - \frac{x^2}{\sin^2\operatorname{am}\frac{(n-1)K'}{n}}\right)}{\left(1 - k'^2x^2\sin^2\operatorname{am}\frac{2K'}{n}\right)\left(1 - k'^2x^2\sin^2\operatorname{am}\frac{4K'}{n}\right)\cdots\left(1 - k'^2x^2\sin^2\operatorname{am}\frac{(n-1)K'}{n}\right)} \quad (\text{mod. } k')$$

$$(cc) \quad z = \sqrt{\frac{{\lambda_1'}^n}{k'}} \sin\operatorname{am}\frac{u}{M_1} \sin\operatorname{am}\left(\frac{u}{M_1} + \frac{4i\Lambda_1}{n}\right) \sin\operatorname{am}\left(\frac{u}{M_1} + \frac{8i\Lambda_1}{n}\right) \cdots \sin\operatorname{am}\left(\frac{u}{M_1} + \frac{4(n-1)i\Lambda_1}{n}\right) \quad (\text{mod. } \lambda_1')$$

$$= \frac{nM_1 y\left(1 + \frac{y^2}{\tan^2\operatorname{am}\frac{2\Lambda_1}{n}}\right)\left(1 + \frac{y^2}{\tan^2\operatorname{am}\frac{4\Lambda_1}{n}}\right)\cdots\left(1 + \frac{y^2}{\tan^2\operatorname{am}\frac{(n-1)\Lambda_1}{n}}\right)}{\left(1 + {\lambda_1'}^2y^2\tan^2\operatorname{am}\frac{2\Lambda_1}{n}\right)\left(1 + {\lambda_1'}^2y^2\tan^2\operatorname{am}\frac{4\Lambda_1}{n}\right)\cdots\left(1 + {\lambda_1'}^2y^2\tan^2\operatorname{am}\frac{(n-1)\Lambda_1}{n}\right)} \quad (\text{mod. } \lambda_1)$$

$$\Lambda_1 = \frac{K}{M_1}; \quad \Lambda_1' = \frac{K'}{nM_1}$$

1.17 General analytical Formulas for the Multiplication of elliptic Functions

28.

Using two supplementary transformations it is possible to construct formulas for the multiplication or formulas by means of which the elliptic functions of the argument nu are expressed by elliptic functions of the argument u. To illustrate this with an example let us compose the multiplication from the first transformation and its supplementary one. For this purpose, recall the formula:

$$\sin\operatorname{am}\left(\frac{u}{M}, \lambda\right) = (-1)^{\frac{n-1}{2}} \sqrt{\frac{k^n}{\lambda}} \sin\operatorname{am}\left(u + \frac{4K}{n}\right) \sin\operatorname{am}\left(u + \frac{8K}{n}\right) \cdots \sin\operatorname{am}\left(u + \frac{4(n-1)K}{n}\right)$$

which can also be represented in this way:

$$(-1)^{\frac{n-1}{2}} \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right) = \sqrt{\frac{k^n}{\lambda}} \prod \sin\operatorname{am}\left(u + \frac{2mK}{n}\right)$$

while m denotes the numbers $0, \pm 1, \pm 2, \cdots, \pm\frac{n-1}{2}$. In this formula let us put $u + \frac{2m'iK'}{n}$ instead of u, whence $\frac{u}{M}$ goes over into $\frac{u}{M} + \frac{2m'iK'}{nM} = \frac{u}{M} + \frac{2m'i\Lambda'}{n}$: This yields:

$$(-1)^{\frac{n-1}{2}} \sin\operatorname{am}\left(\frac{u}{M} + \frac{2m'i\Lambda'}{n}, \lambda\right) = \sqrt{\frac{k^n}{\lambda}} \prod \sin\operatorname{am}\left(u + \frac{2mK + 2m'iK'}{n}\right).$$

Now, if also the values $0, \pm 1, \pm 2, \cdots, \pm\frac{n-1}{2}$ are assigned to m', so that both m and m' take on these values, after having taken the product we obtain:

$$(-1)^{\frac{n-1}{2}}\prod \sin\operatorname{am}\left(\frac{u}{M}+\frac{2m'i\Lambda'}{n},\lambda\right)=\sqrt{\frac{k^{nn}}{\lambda}}\prod \sin\operatorname{am}\left(u+\frac{2mK+2m'iK'}{n}\right),$$

where in the one product m', in the other both m and m' take on all the values $0, \pm 1, \pm 2, \cdots, \pm\frac{n-1}{2}$.

But we saw in the preceding § that it is:

$$\sin\operatorname{am}(nu,k)=\sqrt{\frac{\lambda^n}{k}}\sin\operatorname{am}\frac{u}{M}\sin\operatorname{am}\left(\frac{u}{M}+\frac{4i\Lambda'}{n}\right)\sin\operatorname{am}\left(\frac{u}{M}+\frac{8i\Lambda'}{n}\right)\cdots\sin\operatorname{am}\left(\frac{u}{M}+\frac{4(n-1)i\Lambda'}{n}\right)\ (\text{mod. }\lambda),$$

which formula can also be represented this way:

$$(1.)\qquad \sin\operatorname{am}(nu,k)=\sqrt{\frac{\lambda^n}{k}}\prod \sin\operatorname{am}\left(\frac{u}{M}+\frac{2m'i\Lambda'}{n},\lambda\right)$$

In the same way one finds:

$$(2.)\qquad \cos\operatorname{am} nu=\sqrt{\left(\frac{k}{k'}\right)^{nn-1}}\prod \cos\operatorname{am}\left(u+\frac{2mK+2m'iK'}{n}\right)$$

$$(3.)\qquad \Delta\operatorname{am} nu=\sqrt{\left(\frac{1}{k'}\right)^{nn-1}}\prod \Delta\operatorname{am}\left(u+\frac{2mK+2m'iK'}{n}\right).$$

These formulas are easily reduced to this form:

$$(4.)\qquad \sin\operatorname{am} nu=\sin\operatorname{am} u\prod\frac{1-\frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{2mK+2m'iK'}{n}}}{1-k^2\sin^2\operatorname{am}\frac{2mK+2m'iK'}{n}\sin^2\operatorname{am} u}$$

$$(5.)\qquad \cos\operatorname{am} nu=\sin\operatorname{am} u\prod\frac{1-\frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{2mK+2m'iK'}{n}}}{1-k^2\sin^2\operatorname{am}\frac{2mK+2m'iK'}{n}\sin^2\operatorname{am} u}$$

$$(6.)\qquad \Delta\operatorname{am} nu=n\Delta\operatorname{am} u\prod\frac{1-k^2\sin^2\operatorname{coam}\frac{2mK+2m'iK'}{n}\sin^2\operatorname{am} u}{1-k^2\sin^2\operatorname{am}\frac{2mK+2m'iK'}{n}\sin^2\operatorname{am} u}$$

It is convenient to add the following:

$$
\begin{aligned}
&(7.) \qquad \prod \sin^2 \operatorname{am} \frac{2mK + 2m'iK'}{n} &&= \frac{(-1)^{\frac{n-1}{2}} n}{k^{\frac{nn-1}{2}}} \\
&(8.) \qquad \prod \cos^2 \operatorname{am} \frac{2mK + 2m'iK'}{n} &&= \left(\frac{k'}{k}\right)^{\frac{nn-1}{2}} \\
&(9.) \qquad \prod \Delta \operatorname{am} \frac{2mK + 2m'iK'}{n} &&= k'^{\frac{nn-1}{2}}
\end{aligned}
$$

In the last six formulas the number m only takes on the positive values 0, 1, 2, 3, $\cdots$, $\frac{n-1}{2}$, nevertheless in such a way that, if $m = 0$, only the positive values 0, 1, 2, 3, $\cdots$, $\frac{n-1}{2}$ are assigned also to m'. These and other formulas for the multiplications were already published first by Abel in a different but equivalent form, whence we were able to abbreviate on this subject.

1.18 On the Properties of modular Equations

29.

Since λ depends on k in the same way as k depends on λ_1 and λ'_1 on k' in the same way as k' on λ': It is clear, if one constructs sequences of moduli which can be transformed into each other according to the same law where the one sequence contains the modulus k, the other its complement k' that in them the terms will occur in the same order one after another:

$$
\begin{gathered}
\cdots, \lambda, \;\; k, \;\; \lambda_1, \;\; \cdots \\
\cdots, \lambda'_1, k', \lambda', \;\; \cdots,
\end{gathered}
$$

this was already observed and proved by direct calculation in the transformations of second and third order first by Legendre. Since similar results are true for all transformed and imaginary moduli it is clear, while λ denotes any arbitrary transformed modulus, that the algebraic equations formed between k and λ or any between $u = \sqrt[4]{k}$ and $u = \sqrt[4]{v}$ which we called *modular equations* are not changed,

1.) if k and λ are interchanged
2.) if k' is put instead of k, λ' instead of λ.

we already observed this in the modular equations which belong to the transformations of third and fifth order:

$$\begin{aligned} &(1.) \qquad u^4 - v^4 + 2uv(1 - u^2v^2) = 0 \\ &(2.) \qquad u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0, \end{aligned}$$

and by means of clever observations exhibited algebraic formulas for the supplementary transformations. To also check the others by examples, let us transform those equations into others between $kk = u^8$ and $\lambda\lambda = v^8$ which is not possible without long calculations. Having done them one obtains the equations:

$$\begin{aligned} &(1.) \qquad (k^2 - \lambda^2)^4 = 128k^2\lambda^2(1 - k^2)(1 - \lambda^2)(2 - k^2 - \lambda^2 + 2k^2\lambda^2) \\ &(2.) \qquad (k^2 - \lambda^2)^4 = 512k^2\lambda^2(1 - k^2)(1 - \lambda^2)(L - L'k^2 + L''k^4 - L'''k^6), \end{aligned}$$

if in the second it is put:

$$\begin{aligned} L &= 128 - 192\lambda^2 + 78\lambda^4 - 7\lambda^6 \\ L' &= 192 + 252\lambda^2 - 423\lambda^4 - 78\lambda^6 \\ L'' &= 78 + 423\lambda^2 - 252\lambda^4 - 192\lambda^6 \\ L''' &= 7 - 78\lambda^2 + 192\lambda^4 - 128\lambda^6 \end{aligned}$$

These equations go over into a much more convenient form by introducing $q = 1 - 2k^2$, $l = 1 - 2\lambda^2$. Having done this the propounded equations go over into these:

$$\begin{aligned} (1.) \quad (q - l)^4 &= 64\ (1 - q^2)(1 - l^2)[3 + ql] \\ (2.) \quad (q - l)^6 &= 256(1 - q^2)(1 - l^2)[16ql(9 - ql)^2 + 9(45 - ql)(q - l)^2] \\ &= 256(1 - q^2)(1 - l^2)[405(q^2 + l^2) + 486ql - 9ql(q^2 + l^2) - 270q^2l^2 + 16q^3l^3]. \end{aligned}$$

These equations, if k' is put instead of k, λ' instead of λ, whence q goes over into $-q$, l into $-l$, remain unchanged, which was to be proved.

Corollary. Since we saw the propounded modular equations between $q = 1 - 2k^2$ and $l = 1 - 2\lambda^2$ to take on a sufficiently convenient form it can also be interesting to expand the functions H and K' into a power series of the quantity q. This happens beautifully by the series:

$$\begin{aligned}
K\ &= J\left(1+\frac{q^2}{2\cdot 4}+\frac{5\cdot 5\cdot q^4}{2\cdot 4\cdot 6\cdot 8}+\frac{5\cdot 5\cdot 9\cdot 9\cdot q^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\cdots\right)\\
&\quad -\frac{\pi}{2J}\left(\frac{q}{2}+\frac{3\cdot 3\cdot q^3}{2\cdot 4\cdot 6}+\frac{3\cdot 3\cdot 7\cdot 7q^5}{2\cdot 4\cdot 6\cdot 8\cdot 10}+\frac{3\cdot 3\cdot 7\cdot 7\cdot 11\cdot 11\cdot q^7}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\cdots\right)\\
K' &= J\left(1+\frac{q^2}{2\cdot 4}+\frac{5\cdot 5\cdot q^4}{2\cdot 4\cdot 6\cdot 8}+\frac{5\cdot 5\cdot 9\cdot 9\cdot q^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\cdots\right)\\
&\quad +\frac{\pi}{2J}\left(\frac{q}{2}+\frac{3\cdot 3\cdot q^3}{2\cdot 4\cdot 6}+\frac{3\cdot 3\cdot 7\cdot 7q^5}{2\cdot 4\cdot 6\cdot 8\cdot 10}+\frac{3\cdot 3\cdot 7\cdot 7\cdot 11\cdot 11\cdot q^7}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\cdots\right)
\end{aligned}$$

where for the sake of brevity it was put:

$$\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}} = J.$$

In an easier task the equation for the transformation of third order:

$$u^4 - v^4 + 2uv(1-u^2v^2) = 0$$

can be transformed in such a way that the correlation among the modulus and the complements becomes obvious. For, from that equation we obtain:

$$\begin{aligned}
(1-u^4)(1+v^4) &= 1-u^4v^4+2uv(1-u^2v^2) = (1-u^2v^2)(1+uv)^2\\
(1+u^4)(1-v^4) &= 1-u^4v^4-2uv(1-u^2v^2) = (1-u^2v^2)(1-uv)^2,
\end{aligned}$$

having multiplied which equations it results:

$$(1-u^8)(1-v^8) = (1-u^2v^2)^4.$$

Now let:

$$\begin{aligned}
1-u^8 &= k'k' \ = u'^8\\
1-v^8 &= \lambda'\lambda' = v'^8;
\end{aligned}$$

having extracted the square root it is:

$$u'^2 v'^2 = 1 - u^2 v^2,$$

or:

$$u^2 v^2 + u'^2 v'^2 = \sqrt{k\lambda} + \sqrt{k'\lambda'} = 1,$$

which most elegant formulas were already exhibited Legendre. And the formula is proved very elegantly by means of our analytical formulas, from which it follows in the case $n = 3$:

$$\lambda = k^3 \sin^4 \operatorname{coam} 4\omega, \quad \lambda' = \frac{k'^3}{\Delta^4 \operatorname{am} 4\omega},$$

whence:

$$\begin{aligned} \sqrt{k\lambda} \quad &= k^2 \sin^2 \operatorname{coam} 4\omega = \frac{k^2 \cos \operatorname{am} 4\omega}{\Delta^2 \operatorname{am} 4\omega} \\ \sqrt{k'\lambda'} &= \frac{k'^2}{\Delta \operatorname{am} 4\omega}, \end{aligned}$$

whence, because it is:

$$k'k' + kk \cos^2 \operatorname{am} 4\omega = 1 - kk \sin^2 \operatorname{am} 4\omega = \Delta^2 \operatorname{am} 4\omega$$

we obtain, what was to be demonstrated:

$$\sqrt{k\lambda} + \sqrt{k'\lambda'} = 1.$$

To find a simpler equation among u, v, u', v' in the second example, I proceed as follows. I exhibit the propounded equation:

$$u^6 - v^6 + 5u^2 v^2 (u^2 - v^2) + 4uv(1 - u^4 v^4) = 0$$

as follows:

$$(u^2 - v^2)(u^4 + 6u^2 v^2 + v^4) + 4uv(1 - u^4 v^4) = 0,$$

which is easily seen to take on the following two forms:

$$
\begin{aligned}
(u^2 - v^2)(u+v)^4 &= -4uv(1-u^4)(1+v^4)\\
(u^2 - v^2)(u-v)^4 &= -4uv(1+u^4)(1-v^4),
\end{aligned}
$$

having multiplied which equations it results:

$$(u^2 - v^2)^6 = 16u^2v^2(1-u^8)(1-v^8) = 16u^2v^2u'^8v'^8.$$

Because at the same time, as it was proved above, u^8 and v^8 go over into u'^8 and v'^8, respectively, we also obtain:

$$(v'^2 - u'^2)^6 = 16u'^2v'^2(1-u'^8)(1-v'^8) = 16u'^2v'^2u^8v^8.$$

Hence having done the division and extracted the roots it is found:

$$\frac{u^2 - v^2}{v'^2 - u'^2} = \frac{u'v'}{uv} \quad \text{or} \quad uv(u^2 - v^2) = u'v'(v'^2 - u'^2)$$

or:

$$\sqrt[4]{k\lambda} = (\sqrt{k} - \sqrt{\lambda}) = \sqrt[4]{k'\lambda'}(\sqrt{\lambda'} - \sqrt{k'}).$$

31.

There is even another extraordinary property of modular equations:

$$
\begin{aligned}
&u^4 - v^4 + 2uv(1-u^2v^2) = 0\\
&u^6 - v^6 + 5u^2v^2(u^2-v^2) + 4uv(1-u^4v^4) = 0
\end{aligned}
$$

which is seen on first sight, of course that they remain unchanged, if instead of u, v one puts $\frac{1}{u}$, $\frac{1}{v}$. To show this in general for modular equations, let us note the following things, which can also be of use for other questions.

If one puts $y = kx$, one obtains:

$$\frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = \frac{kdx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

whence, since at the same time $x = 0$ as $y = 0$:

$$\int_0^y \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = k\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Hence having put

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = u,$$

it is:

$$\int_0^y \frac{dy}{\sqrt{(1-y^2)(1-\frac{y^2}{k^2})}} = ku,$$

whence $x = \sin\operatorname{am}(u,k)$, $y = \sin\operatorname{am}\left(ku, \frac{1}{k}\right)$. Hence the following equation results:

$$\sin\operatorname{am}\left(ku, \frac{1}{k}\right) = k\sin\operatorname{am}(u,k),$$

whence also:

$$\begin{aligned}
\cos\operatorname{am}\left(ku, \frac{1}{k}\right) &= \Delta\operatorname{am}(u,k)\\
\Delta\operatorname{am}\left(ku, \frac{1}{k}\right) &= \cos\operatorname{am}(u,k)\\
\tan\operatorname{am}\left(ku, \frac{1}{k}\right) &= \frac{k}{k'}\cos\operatorname{coam}(u,k)\\
\sin\operatorname{coam}\left(ku, \frac{1}{k}\right) &= \frac{1}{\sin\operatorname{coam}(u,k)}\\
\cos\operatorname{coam}\left(ku, \frac{1}{k}\right) &= ik'\tan\operatorname{am}(u,k)\\
\Delta\operatorname{coam}\left(ku, \frac{1}{k}\right) &= \frac{ik'}{\cos\operatorname{am}(u,k)}\\
\tan\operatorname{coam}\left(ku, \frac{1}{k}\right) &= \frac{-i}{\cos\operatorname{coam}(u,k)}
\end{aligned}$$

Further, by putting iu instead of u, because the complement of the modulus $\frac{1}{k}$ becomes $\frac{ik'}{k}$, using the formulas in § 19 we obtain:

$$\begin{aligned}
\sin\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \cos\operatorname{coam}(u, k') \\
\cos\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \sin\operatorname{coam}(u, k') \\
\Delta\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \frac{1}{\Delta\operatorname{coam}(u, k')} \\
\tan\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \cot\operatorname{coam}(u, k') \\
\sin\operatorname{coam}\left(ku, \frac{ik'}{k}\right) &= \cos\operatorname{am}(u, k') \\
\cos\operatorname{coam}\left(ku, \frac{ik'}{k}\right) &= \sin\operatorname{am}(u, k') \\
\sin\operatorname{am}\left(ku, \frac{ik'}{k}\right) &= \cos\operatorname{coam}(u, k') \\
\Delta\operatorname{coam}\left(ku, \frac{ik'}{k}\right) &= \frac{\Delta\operatorname{am}(u, k')}{k} \\
\tan\operatorname{coam}\left(ku, \frac{ik'}{k}\right) &= \cot\operatorname{am}(u, k').
\end{aligned}$$

Now, let us investigate, what becomes of K, K' or $\arg.\operatorname{am}\left(\frac{\pi}{2}, k\right)$, $\arg.\operatorname{am}\left(\frac{\pi}{2}, k'\right)$, if one puts $\frac{1}{k}$ instead of k; or let us investigate the values of the expressions $\arg.\operatorname{am}\left(\frac{\pi}{2}, k\right)$, $\arg.\operatorname{am}\left(\frac{\pi}{2}, k'\right)$, which in the notation used by Legendre would be: $F^1\left(\frac{1}{k}\right)$, $F^1\left(\frac{ik'}{k}\right)$. But, at first it is:

$$\arg.\operatorname{am}\left(\frac{\pi}{2}, \frac{1}{k}\right) = \int\limits_0^1 \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = \int\limits_0^k \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} + \int\limits_k^1 \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}}.$$

Having put $y = kx$ it is:

$$\int\limits_0^k \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = k\int\limits_0^1 \frac{dx}{\sqrt{(1-y^2)\left(1-k^2x^2\right)}} = kK.$$

To find the other integral $\int\limits_k^1 \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}}$ let us put $y = \sqrt{1-k'k'x^2}$, whence $\frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = \frac{-kdx}{\sqrt{(1-x^2)(1-k'k'x^2)}}$. Now, because x hence increases from 0 to 1 at the same time as y decreases from 1 to k, we obtain:

$$\int\limits_k^1 \frac{dy}{\sqrt{(1-y^2)\left(1-\frac{y^2}{k^2}\right)}} = -i\int\limits_k^1 \frac{dy}{\sqrt{(1-y^2)\left(\frac{y^2}{k^2}-1\right)}} = i\int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k'k'x^2)}} = -ikK'$$

Hence it results:

$$\arg.\operatorname{am}\left(\frac{\pi}{2},\frac{1}{k}\right) = \left\{\arg.\operatorname{am}\left(\frac{\pi}{2},k\right) - i\arg.\operatorname{am}\left(\frac{\pi}{2},k'\right)\right\} = k\left\{K - iK'\right\},$$

or if k is changed into $\frac{1}{k}$, K goes over into $\{K - iK'\}$.

Secondly, having put $y = \cos\varphi$ it is:

$$\int\limits_0^1 \frac{dy}{\sqrt{(1-y^2)\left(1+\frac{k'k'}{kk}y^2\right)}} = k\int\limits_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k'k'\sin^2\varphi}} = kK',$$

whence it is:

$$\arg.\operatorname{am}\left(\frac{\pi}{2},\frac{ik'}{k}\right) = k\arg.\operatorname{am}\left(\frac{\pi}{2},k'\right) = kK',$$

or if k is changed into $\frac{1}{k}$, K' goes over into kK'.

Therefore, in general, having changed k into $\frac{1}{k}$, $mK + im'K'$ goes over into $k\{mK + (m'-m)iK'\}$, whence $\sin\operatorname{coam}\left\{\frac{kp(mK+(m'-m)iK'}{n},k\right\}$ goes over into $\sin\operatorname{coam}\left\{\frac{p(mK+m'iK'}{n},k\right\}$ which from the formula

$$\sin\operatorname{coam}\left(ku,\frac{1}{k}\right) = \frac{1}{\sin\operatorname{coam}(u,k)}$$

becomes:

$$\sin\operatorname{coam}\left\{\frac{kp(mK+(m'-m)iK'}{n},\frac{1}{k}\right\}=\frac{1}{\sin\operatorname{coam}\left\{\frac{kp(mK+(m'-m)iK'}{n},k\right\}}.$$

Therefore, having put $\omega = \frac{mK+m'iK'}{n} = \omega$, $\frac{mK+(m'-m)iK'}{n} = \omega_1$, the expression:

$$\lambda = k^n[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\sin\operatorname{coam}6\omega\cdots\sin\operatorname{coam}(n-1)\omega]^4,$$

having changed k into $\frac{1}{k}$ goes over into this one:

$$\frac{1}{k^n[\sin\operatorname{coam}2\omega_1\sin\operatorname{coam}4\omega_1\sin\operatorname{coam}6\omega_1\cdots\sin\operatorname{coam}(n-1)\omega_1]^4}=\frac{1}{\mu},$$

where μ itself is a root of a modular equation, or one of the total amount of moduli into which the propounded modulus k can be transformed by means of a transformation of n-th order. For, one of the values which ω can have so that a transformed modulus results will also be ω_1. Hence also the reason is clear, why in general modular equations, having changed k into $\frac{1}{k}$, λ into $\frac{1}{\lambda}$ must remain unchanged.

Additionally, I mention, if according to the same law of transformation k is transformed into $k^{(m)}$, λ into $\lambda^{(m)}$ and if then $k^{(m)}$ is put instead of k, that also λ goes over into $\lambda^{(m)}$; hence modular equations, if k is changed into $k^{(m)}$ at the same time as λ is changed into $\lambda^{(m)}$, have to remain unchanged. So, for the sake of an example, the equation $\sqrt{k\lambda}+\sqrt{k'\lambda'}=1$ which is a modular equation for the transformation of third order, must remain unchanged, if one puts $\frac{1-k'}{1+k'}$, $\frac{1-\lambda'}{1+\lambda'}$, instead of k, λ, respectively, whence one has to put $\frac{2\sqrt{k'}}{1+k'}$, $\frac{2\sqrt{\lambda'}}{1+\lambda'}$, which is known to be achieved by means of a transformation of second order. The equation $\sqrt{k\lambda}+\sqrt{k'\lambda'}=1$ goes over into this one:

$$\sqrt{\frac{(1-k')(1-\lambda')}{(1+k')(1+\lambda')}}+\frac{2\sqrt[4]{k'\lambda'}}{\sqrt{(1+k')(1+\lambda')}}=1,$$

or:

$$2\sqrt[4]{k'\lambda'}=\sqrt{(1+k')(1+\lambda')}-\sqrt{(1-k')(1-\lambda')}.$$

Having squared both sides it results:

$$4\sqrt{k'\lambda'} = 2(1+k'\lambda') - 2k\lambda, \quad \text{or} \quad k\lambda = 1 + k'\lambda' - 2\sqrt{k'\lambda'},$$

which having extracted the roots reduces to the propounded one:

$$\sqrt{k\lambda} = 1 - \sqrt{k'\lambda'} \quad \text{or} \quad \sqrt{k\lambda} + \sqrt{k'\lambda'} = 1.$$

This example was already propounded by Legendre. But, it can be shown in general on the composition of transformations that having used two or several transformations successively that one reaches the same, no matter in which order they are applied.

32.

But among the properties of modular equations there is one which I consider to be most remarkable and outstanding, namely, *that they all satisfy the same differential equation of third order*. For its investigation we will nevertheless have to mention some things in advance.

It is well-known having put:

$$aK + bK' = Q$$

that it will be:

$$k(1-k^2)\frac{d^2Q}{dk^2} + (1-3k^2)\frac{dQ}{dk} = kQ,$$

a, b denoting arbitrary constants. Therefore, having also put:

$$a'K + b'K' = Q',$$

a', b' denoting other arbitrary constants, it will be:

$$k(1-k^2)\frac{d^2Q'}{dk^2} + (1-3k^2)\frac{dQ'}{dk} = kQ'.$$

Having combined these equations one obtains:

$$k(1-k^2)\left\{Q\frac{d^2Q'}{dk^2} - Q'\frac{d^2Q}{dk^2}\right\} + (1-3k^2)\left\{Q\frac{dQ'}{dk} - Q'\frac{dQ}{dk}\right\} = 0$$

whence after an integration:

$$k(1-k^2)\left\{Q\frac{dQ'}{dk}-Q'\frac{dQ}{dk}\right\}=(ab'-a'b)k(1-k^2)\left\{K\frac{dK'}{dk}-K'\frac{dK}{dk}\right\}=(ab'-a'b)C.$$

The constant C was already found from a special case by Legendre to be $=-\frac{\pi}{2}$, whence now it is:

$$Q\frac{dQ'}{dk}-Q'\frac{dQ}{dk}=-\frac{1}{2}\frac{\pi(ab'-a'b)dk}{k(1-k^2)},$$

or

$$d\frac{Q'}{Q}=-\frac{1}{2}\frac{\pi(ab'-a'b)}{k(1-k^2)QQ}$$

Similarly, λ denoting another arbitrary modulus, having put

$$\alpha\Lambda+\beta\Lambda'=L,\qquad \alpha'\Lambda+\beta'\Lambda'=L',$$

it will be

$$d\frac{L}{L'}=-\frac{1}{2}\frac{\pi(\alpha\beta'-\alpha'\beta)d\lambda}{\lambda(1-\lambda\lambda)LL}.$$

Let λ be the modulus into which k is transformed by a first transformation of n-th order; further, let $Q=K$, $Q'=K'$, $K=\Lambda$, $L'=\Lambda'$; it will be:

$$\frac{L'}{L}=\frac{\Lambda'}{\Lambda}=\frac{nK'}{K}=\frac{nQ'}{Q},$$

whence it follows:

$$\frac{ndk}{k(1-k^2)KK}=\frac{d\lambda}{\lambda(1-\lambda\lambda)\Lambda\Lambda}.$$

But for the transformation we found $\Lambda=\frac{K}{nM}$, whence:

$$MM=\frac{1}{n}\cdot\frac{\lambda(1-\lambda^2)dk}{k(1-k^2)d\lambda}.$$

In the second transformation, we saw that $\frac{\Lambda_1'}{\Lambda_1}=\frac{1}{n}\cdots\frac{K'}{K}$, $\Lambda_1=\frac{K}{M_1}$, whence:

$$\frac{dk}{k(1-k^2)KK}=\frac{nd\lambda_1}{\lambda_1(1-\lambda_1^2)\Lambda_1\Lambda_1},$$

whence also here:

$$M_1 M_1 = \frac{1}{n} \cdot \frac{\lambda_1(1-\lambda_1^2)dk}{k(1-k^2)d\lambda_1}.$$

But generally, whatever the modulus λ is, whether real or imaginary, into which the propounded modulus k can be transformed by means of a transformation of n-th order, the following equation will hold:

$$MM = \frac{1}{n} \cdot \frac{\lambda(1-\lambda^2)dk}{k(1-k^2)d\lambda}.$$

To show this, I note that in general one obtains equations of the form:

$$\begin{aligned} \alpha\Lambda \ \ + i\beta\Lambda' &= \frac{aK + ibK'}{nM} \\ \alpha'\Lambda' + i\beta'\Lambda &= \frac{a'K' + ib'K}{nM}, \end{aligned}$$

where a, a', α, α' denote odd numbers, b, b', β, β' denote even numbers, both either positive or negative of such a kind that $aa' + bb' = 1$, $\alpha\alpha' + \beta\beta' = 1$. Hence having put:

$$\begin{aligned} aK + ibK' &= Q, \qquad & a'K' + ib'K &= Q' \\ a\Lambda + ib\Lambda' &= L, \qquad & \alpha'\Lambda' + i\beta'K &= L' \end{aligned}$$

because it is $aa' + bb' = 1$, $\alpha\alpha' + \beta\beta' = 1$, we obtain:

$$d\frac{Q'}{Q} = -\frac{1}{2}\frac{n\pi dk}{k(1-k^2)QQ'}, \qquad d\frac{L'}{L} = -\frac{1}{2}\frac{\pi d\lambda}{\lambda(1-\lambda^2)LL'},$$

whence, because it is:

$$\frac{Q'}{Q} = \frac{L'}{L} \quad L = \frac{Q}{nM},$$

it is in general:

$$MM = \frac{1}{n} \cdot \frac{\lambda(1-\lambda^2)dk}{k(1-k^2)d\lambda}.$$

I mention that the found equation can also be exhibited this way:

$$MM = \frac{1}{n} \cdot \frac{\lambda^2(1-\lambda^2)d(k^2)}{k^2(1-k^2))d(\lambda^2)} = \frac{1}{n} \cdot \frac{\lambda'^2(1-\lambda'^2)d(k'^2)}{k'^2(1-k'^2)d(\lambda'^2)},$$

whence we see that the expression MM is not changed, if one puts k', λ' instead of k, λ, or what we demonstrated above that in complementary transformations, not taking into account the sign, the multiplicator M is the same. Further, by changing k into λ, λ into k, having done which the transformation goes over into the supplementary, MM is changed into

$$\frac{1}{n} \cdot \frac{k(1-k^2)d\lambda}{\lambda(1-\lambda^2)dk} = \frac{1}{nnMM} \quad \text{or} \quad M \quad \text{into} \quad \frac{1}{nM},$$

what was already proved above.

33.

Having put $Q = aK + ibK'$, $L = \alpha\Lambda + i\beta\Lambda'$ it is always possible to determine constants a, b, α, β such that $L = \frac{Q}{M}$ or $Q = ML$. Further, one has the equations:

$$\begin{aligned} &(1.) \qquad (k-k^3)\frac{d^2Q}{dk^2} + (1-3k^2)\frac{dQ}{dk} - kQ = 0 \\ &(2.) \qquad (\lambda-\lambda^3)\frac{d^2Q}{d\lambda^2} + (1-3\lambda^2)\frac{dQ}{d\lambda} - \lambda Q = 0, \end{aligned}$$

which can also be represented this way:

$$\begin{aligned} &(3.) \qquad \frac{d}{dk}\left\{\frac{(k-k^3)dQ}{dk}\right\} - kQ = 0, \\ &(4.) \qquad \frac{d}{d\lambda}\left\{\frac{(\lambda-\lambda^3)dL}{d\lambda}\right\} - \lambda L = 0. \end{aligned}$$

Let us in the equation:

$$(k-k^3)\frac{d^2Q}{dk^2} + (1-3k^2)\frac{dQ}{dk} - kQ = 0,$$

put $Q = ML$, it results:

$$L\left\{(k-k^3)\frac{d^2M}{dk^2}+(1-3k^2)\frac{dM}{dk}-kM\right\}$$
$$+\frac{dL}{dk}\left\{2(k-k^3)\frac{dM}{dk}+(1-3k^2)M\right\}+(k-k^3)M\frac{d^2L}{d^2k}=0,$$

having multiplied which equation by M, we obtain:

$$(5.)\qquad LM\left\{(k-k^3)\frac{d^2M}{dk^2}+(1-3k^2)\frac{dM}{dk}-kM\right\}+\frac{d}{dk}\left\{\frac{(k-k^3)MMdL}{dk}\right\}=0.$$

But from the preceding § it is:

$$MM=\frac{(\lambda-\lambda^3)dk}{n(k-k^3)d\lambda},\quad \text{whence}\quad \frac{(k-k^3)MMdL}{dk}=\frac{(\lambda-\lambda^3)dL}{nd\lambda}.$$

Further, from equation (4.) it is:

$$d\left\{\frac{(\lambda-\lambda^3)}{d\lambda}\right\}=\lambda Ld\lambda,$$

whence it follows:

$$\frac{d}{dk}\left\{\frac{(k-k^3)MMdL}{dk}\right\}=\frac{1}{n}\frac{d}{dk}\left\{\frac{(\lambda-\lambda^3)}{d\lambda}\right\}=\frac{\lambda Ld\lambda}{ndk}.$$

Hence equation (5.), having divided it by L, goes over into this one:

$$(6.)\qquad M\left\{(k-k^3)\frac{d^2M}{dk^2}+(1-3k^2)\frac{dM}{dk}-kM\right\}=\frac{\lambda d\lambda}{ndk}=0.$$

If the value of M from the equation $MM=\frac{(\lambda-\lambda^3)dk}{n(k-k^3)d\lambda}$ is substituted in this equation, one obtains a differential equation for the moduli k and λ, which equation is easily seen to ascend to third order. After the cumbersome calculation it is found:

$$(7.)\qquad \frac{3d^2\lambda^2}{dk^4}-\frac{2d\lambda}{dk}\cdot\frac{d^3\lambda}{dk^3}+\frac{d\lambda^2}{dk^2}\left\{\left[\frac{1+k^2}{k-k^3}\right]^2+\left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\frac{d\lambda^2}{dk^2}\right\}=0.$$

In this equation dk is considered as a constant differential. If it pleases to transform it into another in which no differential is assumed to be constant, one will have to put:

$$\frac{d^2\lambda}{dk^2} = \frac{d^2\lambda}{dk^2} - \frac{d\lambda d^2k}{dk^3}$$
$$\frac{d^3\lambda}{dk^3} = \frac{d^3\lambda}{dk^3} - \frac{3d^2\lambda d^2k}{dk^4} - \frac{d\lambda d^3k}{dk^4} + \frac{3d\lambda d^2k^2}{dk^5}$$

whence it follows:

$$\frac{3d^2\lambda^2}{dk^4} - \frac{2d\lambda}{dk}\frac{d^3\lambda}{dk^3} = \frac{3d^2\lambda^2}{dk^4} - \frac{3d\lambda^2d^2k^2}{dk^6} + \frac{2d\lambda^2d^3k}{dk^5} - \frac{2d\lambda d^3\lambda}{dk^4}.$$

Hence equation (7.) multiplied by dk^6 goes over into the following in which no differential is assumed to be constant, or in which any arbitrary one can be considered as such:

$$(8.)\quad 3\left\{dk^2d^2\lambda^2 - d\lambda^2d^2k^2\right\} - 2dkd\lambda\left\{dkd^3\lambda - d\lambda d^3k\right\} + dk^2d\lambda^2\left\{\left[\frac{1+k^2}{k-k^3}\right]^2 dk^2 - \left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2 d\lambda^2\right\} = 0.$$

This equation, having interchanged the variables λ and k, is obvious to remain unchanged, what we proved above on modular equations.

It is worth one's while to investigate our differential equation of third order by another method. For this aim, let us introduce the quantity:

$$(k-k^3)QQ = s.$$

into the equation from which we start:

$$(k-k^3)\frac{d^2Q}{dk^2} + (1-3k^2)\frac{dQ}{dk} - kQ = 0$$

It is

$$\frac{ds}{dk} = (1-3k^2)QQ + 2(k-k^3)Q\frac{dQ}{dk}$$
$$\frac{d^2s}{dk^2} = -6kQQ + 4(1-3k^2)Q\frac{dQ}{dk} + 2(k-k^3)\left[\frac{dQ}{dk}\right]^2 + 2(k-k^3)Q\frac{d^2}{dk^2}.$$

If in the equation one puts:

$$(k-k^3)\frac{d^2Q}{dk^2} = kQ - (1-3k^2)\frac{dQ}{dk},$$

it results:

$$\begin{aligned}\frac{d^2s}{dk^2} &= -4kQQ + 2(1-3k^2)Q\frac{dQ}{dk} + 2(k-k^3)\left[\frac{dQ}{dk}\right]^2 \\ &= 2\frac{dQ}{dk}\left\{(1-3k^2)Q + (k-k^3)\frac{dQ}{dk}\right\} - 4kQQ.\end{aligned}$$

Having multiplied this equation by $2s = 2(k-k^3)QQ$ one obtains:

$$\frac{2sd^2s}{dk^2} = 2(k-k^3)Q\frac{dQ}{dk}\left\{2(1-3k^2)QQ + 2(k-k^3)Q\frac{dQ}{dk}\right\} - 8k^2(1-k^2)Q^4,$$

or because it is:

$$\begin{aligned}2(k-k^3)Q\frac{dQ}{dk} &= \frac{ds}{dk} - (1-3k^2)QQ \\ 2(1-3k^2)QQ + 2(k-k^3)Q\frac{dQ}{dk} &= \frac{ds}{dk} + (1-3k^2)QQ,\end{aligned}$$

we obtain:

$$\frac{2sd^2s}{dk^2} = \left[\frac{ds}{dk}\right]^2 - (1-3k^2)^2Q^4 - 8k^2(1-k^2)Q^4 = \left[\frac{ds}{ds}\right]^2 - (1+k^2)^2Q^4,$$

or

$$(9.)\qquad \frac{2sd^2s}{dk^2} - \left[\frac{ds}{dk}\right]^2 + \left[\frac{1+k^2}{k-k^3}\right]^2 ss = 0.$$

But having put $a'K + b'K' = Q'$, $\frac{Q'}{Q} = t$ we see that $\frac{dt}{dk} = \frac{m}{(k-k^3)QQ} = \frac{m}{s}$, where m denotes a constant whence $s = \frac{mdk}{dt}$. Let us transform equation (9.) into another in which dt is assumed to be constant. It will be $\frac{ds}{dk} = \frac{md^2k}{dtdk}$, $\frac{d^2}{dk^2} = \frac{md^3k}{dtdk^2} - \frac{md^2k^2}{dtdk^3}$; having substituted these from equation (9.) it results:

$$\frac{2d^3k}{dt^2dk} - \frac{3d^2k^2}{dt^2dk^2} + \left[\frac{1+k^2}{k-k^3}\right]^2\frac{dk^2}{dt^2} = 0,$$

or

$$(10.) \qquad 2d^3kdk - 3d^2k^2 + \left[\frac{1+k^2}{k-k^3}\right]^2 \frac{dk^2}{dk^4} = 0,$$

where is to be differentiated with respect to t which went out of the equation.

By putting $\frac{\alpha'\Lambda+\beta'\Lambda'}{\alpha\Lambda+\beta\Lambda^{\beta}} = \omega$, one will be able to determine constants α, β, α', β', if λ is the transformed modulus, in such a way that $t = \omega$; and in like manner we obtain:

$$(11.) \qquad 2d^3\lambda d\lambda - 3d^2\lambda^2 + \left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2 d\lambda^4 = 0,$$

in which equation it is to be differentiated with respect to $t = \omega$. Multiply equation (10.) by $d\lambda^2$, equation (11.) by dk^2; after the substitution one obtains:

$$(12.)\ 2dkd\lambda \left\{d\lambda d^3k - dkd^3\lambda\right\} - 3\left\{d\lambda^2 d^2k^2 - dk^2 d^2\lambda^2\right\} + dk^2 d\lambda^2 \left\{\left[\frac{1+k^2}{k-k^3}\right]^2 dk^2 - \left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2 d\lambda^2\right\} = 0.$$

But this equations agrees with equation (8.) in which we know that any arbitrary differential can be considered as constant and even though having done the substitution it was found that dt is a constant differential it will also hold whatever other differential is considered as a constant.

So, lo and behold this differential equation of third order which nevertheless has innumerable particular solutions and some of those particular solutions are those we called modular equations. But the complete integral depends on elliptic functions which is $t = \omega$ or

$$\frac{a'K + bK'}{aK + bK'} = \frac{\alpha'\Lambda + \beta'\Lambda'}{\alpha\Lambda + \beta\Lambda'},$$

which equation can also be represented this way:

$$mK\Lambda + m'K'\Lambda' + m''K\Lambda' + m'''K'\Lambda = 0,$$

m, m', m'', m''' denoting arbitrary constants. This integration is to be considered to be of highest depth.

We could investigate whether modular equations for the transformations of third and fifth order indeed (and they have to) satisfy our differential equation of third order. But because this seems to demand too long calculations, it

shall suffice to show the same for the transformation of second order, where $\lambda = \frac{1-k'}{1+k'}$.
Let us consider dk' to be constant, it is:

$$\begin{aligned}
\lambda &= \frac{1-k'}{1+k'} = -1 + \frac{2}{1+k'} & k^2 + k'^2 &= 1 \\
\frac{d\lambda}{dk'} &= \frac{-2}{(1+k')^2} & \frac{dk}{dk'} &= -\frac{-k'}{k} \\
\frac{d^2\lambda}{dk'^2} &= \frac{4}{(1+k')^3} & \frac{d^2k}{dk'^2} &= -\frac{-1}{k} - \frac{k'^2}{k^3} = \frac{-1}{k^3} \\
\frac{d^3\lambda}{dk'^3} &= \frac{-12}{(1+k')^3} & \frac{d^3k}{dk'^3} &= -\frac{-3k'}{k^5}
\end{aligned}$$

Hence it is:

$$\begin{aligned}
\frac{dk^2d^2\lambda^2 - d\lambda^2d^2k^2}{dk'^6} &= \frac{16k'^2}{k^2(1+k')^6} - \frac{4}{k^6(1+k')^4} \\
=\frac{4[4k^4k'^2 - (1+k')^2]}{k^6(1+k')^6} &= \frac{4[4k'^2(1-k')-1]}{k^6(1+k')^4}.
\end{aligned}$$

Further, it is obtained:

$$\begin{aligned}
\frac{dkd^3\lambda - d\lambda d^3k}{dk'^4} &= \frac{12k'}{k(1+k')^4} - \frac{6k'}{k^5(1+k')^2} = \frac{6k'[2(1-k')^2-1]}{k^5(1+k')^2} \\
\frac{dkd\lambda[dkd^3\lambda - d\lambda d^3k]}{dk'^6} &= \frac{12k'^2[2(1-k')-1]}{k^6(1+k')^4},
\end{aligned}$$

whence it follows:

$$\frac{3[dk^2d^2\lambda^2 - d\lambda^2d^2k^2] - 2dkd\lambda[dkd^3\lambda - d\lambda d^3k]}{dk'^6} = \frac{12(2k'^2-1)}{k^6(1+k')^4}.$$

Further, it is

$$\left[\frac{1+k}{k-k^3}\right]^2 \frac{dk^2}{dk'^2} = \frac{(1+k'^2)^2}{k^4k'^2}$$

$$\left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\frac{d\lambda^2}{dk'^2}=\frac{4}{(1+k')^4}\left[\frac{1+k'}{1-k'}\right]^2\left[\frac{1+k'^2}{2k'}\right]^2=\frac{(1+k'^2)^2}{k'^2k^4},$$

whence

$$\left[\frac{1+k^2}{k-k^3}\right]^2\frac{dk^2}{dk'^2}-\left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\frac{d\lambda^2}{dk'^2}=\frac{3(1-2k'^2)}{k^4k'^2}$$

$$\frac{dk^2d\lambda^2}{dk'^4}=\left\{\left[\frac{1+k^2}{k-k^3}\right]^2\frac{dk^2}{dk'^2}-\left[\frac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\frac{d\lambda^2}{dk'^2}\right\}=\frac{12(1-2k'^2)}{k^6(1+k')^4}.$$

Hence it finally is and what is to be proved

$$\left.\begin{array}{l}\dfrac{3[dk^2d^2\lambda^2-d\lambda^2d^2k^2]-2dkd\lambda[dkd^3\lambda-d\lambda d^3k]}{dk'^6}\\ +\dfrac{dk^2d\lambda^2}{dk'^4}\left\{\left[\dfrac{1+k^2}{k-k^3}\right]^2-\left[\dfrac{1+\lambda^2}{\lambda-\lambda^3}\right]^2\dfrac{d\lambda}{dk'^2}\right\}\end{array}\right\}=\frac{12(2k'^2-1)}{k^6(1+k')^4}+\frac{12(1-2k'^2)}{k^6(1+k')^4}=0.$$

If there would be a finished theory, if a differential equation has algebraic solutions, to find them all, from the propounded differential equation we could find all modular equations which belong to the single orders of transformation. Nevertheless, I know no one who has tackled this difficult matter worth of the analysts' attention, except for Condorcet.

34.

The equation found above

$$MM=\frac{1}{n}\cdot\frac{\lambda(1-\lambda^2)}{k(1-k^2)}\cdot\frac{dk}{d\lambda},$$

by means of which it is possible to also find the quantity M immediately from the found modular equation, seems to be one's while to consider it a little bit more. It is not clear on first sight, how the values of the quantity M agree with the equation found in the transformations of third and fifth order. Therefore, let us consider this more accurately.

a) In the transformation of *third* order having put $u=\sqrt[4]{k}$, $v=\sqrt[4]{\lambda}$ we find:

$$(1.)\qquad u^4-v^4+2uv(1-u^2v^2)=0,$$

which equation we also exhibited this way (§ 16):

$$(2.)\qquad \left(\frac{v+2u^3}{v}\right)\left(\frac{u-2v^3}{u}\right) = -3.$$

Further, we saw that it is:

$$(3.)\qquad M = \frac{v}{v+2u^3} = \frac{2v^3-u}{3u}.$$

Having differentiated equation (1.) we obtain:

$$\frac{du}{dv} = \frac{2v^3-u+3u^3v^2}{2u^3+v-3u^2v^3},$$

or having put $\left(\frac{v+2u^3}{v}\right)\left(\frac{2v^3-u}{u}\right)$ instead of 3:

$$(4.)\qquad \frac{du}{dv} = \frac{2v^3-u}{2u^3+v}\cdot\frac{1+uv^2u^2+2u^5v}{1+u^2v^2-2uv^5}.$$

From equation (1.) it follows:

$$\begin{aligned}
1-u^8 &= (1+u^4)[1-v^4+2uv(1-u^2v^2)] \\
&= 1-u^4v^4+u^4-v^4+2uv(1+u^4)(1-u^2v^2) \\
&= 1-u^4v^4+2u^5v(1-u^2v^2) = (1-u^2v^2)(1+u^2v^2+2u^5v).
\end{aligned}$$

The same way one finds:

$$1-v^8 = (1-u^2v^2)(1+u^2v^2-2uv^5),$$

whence:

$$\frac{1-v^8}{1-u^8} = \frac{1+u^2v^2-2uv^5}{1+u^2v^2+2u^5v},$$

or from equation (4.):

$$\frac{1-v^8}{1-u^8}\cdot\frac{du}{dv} = \frac{2v^3-u}{2u^3+v}..$$

Having multiplied this equation by:

$$\frac{v}{3u} = \frac{v^2}{(2u^3+v)(2v^3-u)},$$

it results:

$$\frac{1}{3}\cdot\frac{v(1-v^8)}{u(1-u^8)}\cdot\frac{du}{dv}=\frac{1}{3}\cdot\frac{\lambda(1-\lambda^2)}{k(1-k^2)}\cdot\frac{dk}{d\lambda}=\left[\frac{v}{v+2u^3}\right]^2=MM,$$

Q.D.E.

b) In the transformation of *fifth* order having put $u=\sqrt[4]{k}$, $v=\sqrt[4]{\lambda}$ we found:

$$(1.)\qquad u^6-v^6+5u^2v^2(u^2-v^2)+4uv(1-u^4v^4)=0,$$

which equation we also exhibited in these ways (§§. 16 and 30):

$$(2.)\qquad \frac{u+v^5}{u(1+u^3v)}\cdot\frac{v-u^5}{v(1-uv^3)}=5$$

$$(3.)\qquad (u^2-v^2)^6=16u^2v^2(1-u^8)(1-v^8).$$

Further, we found:

$$(4.)\qquad M=\frac{v(1-uv^3)}{v-u^5}=\frac{u+v^5}{5u(1+u^3v)}.$$

Having differentiated equation (3.) we obtain:

$$6uv(1-u^8)(1-v^8)(udu-vdv)=u(u^2-v^2)(1-u^8)(1-5v^8)dv+v(u^2-v^2)(1-v^8)(1-5u^8)du,$$

Having multiplied equation (1.) by u^4, v^4 one finds:

$$5u^2-u^{10}+v^2-5u^8v^2=(1-u^4v^4)(v^2+5u^2+4u^5v)$$
$$5v^2-v^{10}+u^2-5u^2v^8=(1-u^4v^4)(u^2+5v^2-4uv^5),$$

whence equation (5.) goes over into this one:

$$(6.)\qquad \frac{v(1-v^8)}{u(1-u^8)}\cdot\frac{du}{dv}=\frac{u^2+5v^2-4uv^5}{v^2+5u^2+4u^5v}.$$

Put $u+v^5$, $u+u^4v=B$, $v-u^5=C$, $v-uv^4=D$ such that:

$$\frac{AC}{BC} = 5, \quad \text{or} \quad AC = 5BD,$$
$$\frac{D}{C} = \frac{A}{5B} = M;$$
$$u^2 + 5v^2 - 4uv^5 = uA + 5vD$$
$$v^2 + 5u^2 + 4u^5v = vC + 5uB,$$

it will be:

$$(7.) \qquad \frac{v(1-v^8)}{u(1-u^8)} \cdot \frac{du}{dv} = \frac{uA + 5vD}{vC + 5uB} = \frac{uAB + vAC}{vCD + uAC} \cdot \frac{D}{B}$$
$$= \frac{uB + vC}{vD + UA} \cdot \frac{AB}{BC} = \frac{AD}{BC} = 5MM.$$

For, it is:

$$uB + vC = vD + uA = u^2 + v^2.$$

Hence also:

$$MM = \frac{1}{5} \cdot \frac{v(1-v^8)}{u(1-u^8)} \cdot \frac{du}{dv} = \frac{1}{5} \cdot \frac{\lambda(1-\lambda^2)}{k(1-k^2)} \cdot \frac{dk}{d\lambda}.$$

Q.D.E.

2 Theory of the Expansion of Elliptic Functions

35.

Having propounded a real modulus k smaller than 1 we saw that the modulus

$$\lambda = k^n \left\{ \sin\operatorname{coam} \frac{2K}{n} \sin\operatorname{coam} \frac{4K}{n} \cdots \sin\operatorname{coam} \frac{(n-1)K}{n} \right\}^4$$

into which it is transformed by the first transformation of n-th order, while n grows, rapidly converges to zero, and hence for the limit $n = \infty$ it is $\lambda = 0$. Then, it will be $\Lambda = \frac{\pi}{2}$, $\operatorname{am}(u, \lambda) = u$, whence from the formulas $\Lambda = \frac{K}{nM}$, $\Lambda' = \frac{K'}{M}$ we obtain:

$$nM = \frac{2K}{\pi}, \quad \frac{\Lambda'}{n} = \frac{K'}{nM} = \frac{\pi K'}{2K}.$$

Now, in the formulas for the supplementary transformation of the first in § 27 let us put $\frac{u}{n}$ instead of u and $n = \infty$: $\operatorname{am}\left(\frac{u}{M}, \lambda\right)$ goes over into $\operatorname{am}\left(\frac{u}{nM}, \lambda\right) = \frac{\pi u}{2K}$, $y = \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)$ into $\sin \frac{\pi u}{2K}$, further $\operatorname{am}(nu)$ into $\operatorname{am}(u)$. Hence from those formulas we obtain the following:

$$\sin\operatorname{am} u = \frac{2Ky}{\pi} \cdot \frac{\left(1 - \frac{y^2}{\sin^2 \frac{i\pi K'}{K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{2i\pi K'}{K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{3i\pi K'}{K}}\right)\cdots}{\left(1 - \frac{y^2}{\sin^2 \frac{i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{3i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{5i\pi K'}{2K}}\right)\cdots}$$

$$\cos\operatorname{am} u = \sqrt{1 - y^2} \cdot \frac{\left(1 - \frac{y^2}{\cos^2 \frac{i\pi K'}{K}}\right)\left(1 - \frac{y^2}{\cos^2 \frac{2i\pi K'}{K}}\right)\left(1 - \frac{y^2}{\cos^2 \frac{3i\pi K'}{K}}\right)\cdots}{\left(1 - \frac{y^2}{\sin^2 \frac{i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{3i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{5i\pi K'}{2K}}\right)\cdots}$$

$$\Delta\operatorname{am} u = \frac{\left(1 - \frac{y^2}{\cos^2 \frac{i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\cos^2 \frac{3i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{5i\pi K'}{2K}}\right)\cdots}{\left(1 - \frac{y^2}{\sin^2 \frac{i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{3i\pi K'}{2K}}\right)\left(1 - \frac{y^2}{\sin^2 \frac{5i\pi K'}{2K}}\right)\cdots}$$

$$\sqrt{\frac{1-\sin\operatorname{am} u}{1+\sin\operatorname{am} u}} = \sqrt{\frac{1-y}{1+y}} \cdot \frac{\left(1-\frac{y}{\cos\frac{i\pi K'}{K}}\right)\left(1-\frac{y}{\cos\frac{2i\pi K'}{K}}\right)\left(1-\frac{y}{\cos\frac{3i\pi K'}{K}}\right)\cdots}{\left(1+\frac{y}{\cos\frac{i\pi K'}{K}}\right)\left(1+\frac{y}{\cos\frac{2i\pi K'}{K}}\right)\left(1+\frac{y}{\cos\frac{3i\pi K'}{K}}\right)\cdots}$$

$$\sqrt{\frac{1-k\sin\operatorname{am} u}{1+k\sin\operatorname{am} u}} = \frac{\left(1-\frac{y}{\cos\frac{i\pi K'}{2K}}\right)\left(1-\frac{y}{\cos\frac{3i\pi K'}{2K}}\right)\left(1-\frac{y}{\cos\frac{5i\pi K'}{2K}}\right)\cdots}{\left(1+\frac{y}{\cos\frac{i\pi K'}{2K}}\right)\left(1+\frac{y}{\cos\frac{3i\pi K'}{2K}}\right)\left(1+\frac{y}{\cos\frac{5i\pi K'}{2K}}\right)\cdots}$$

$$\sin\operatorname{am} u = -\frac{\pi y}{kK} \cdot \left(\frac{\cos\frac{i\pi K'}{2k}}{\sin^2\frac{i\pi K'}{2K}-y^2} + \frac{\cos\frac{3i\pi K'}{2k}}{\sin^2\frac{3i\pi K'}{2K}-y^2} + \frac{\cos\frac{5i\pi K'}{2k}}{\sin^2\frac{5i\pi K'}{2K}-y^2} + \cdots\right)$$

$$\cos\operatorname{am} u = \frac{i\pi\sqrt{1-y^2}}{kK} \cdot \left(\frac{\sin\frac{i\pi K'}{2k}}{\sin^2\frac{i\pi K'}{2K}-y^2} - \frac{\sin\frac{3i\pi K'}{2k}}{\sin^2\frac{3i\pi K'}{2K}-y^2} + \frac{\sin\frac{5i\pi K'}{2k}}{\sin^2\frac{5i\pi K'}{2K}-y^2} - \cdots\right).$$

In the following, let us put $e^{\frac{-\pi k'}{K}} = q$, $\frac{\pi u}{2K} = x$, or $u = \frac{2Kx}{\pi}$, whence $y = \sin\frac{\pi u}{2K} = \sin x$; it is:

$$\sin\frac{mi\pi K'}{K} = \frac{q^m - q^{-m}}{2i} = \frac{i(1-q^{2m})}{2q^m}$$
$$\cos\frac{mi\pi K'}{K} = \frac{q^m + q^{-m}}{2} = \frac{1+q^{2m}}{2q^m},$$

whence it is:

$$1-\frac{y^2}{\sin^2\frac{mi\pi K'}{K}} = 1+\frac{4q^{2m}\sin^2 x}{(1-q^{2m})^2} = \frac{1-2q^{2m}\cos 2x+q^{4m}}{(1-q^{2m})^2}$$
$$1-\frac{y^2}{\cos^2\frac{mi\pi K'}{K}} = 1-\frac{4q^{2m}\sin^2 x}{(1+q^{2m})^2} = \frac{1+2q^{2m}\cos 2x+q^{4m}}{(1+q^{2m})^2}$$
$$1\pm\frac{y}{\cos\frac{mi\pi K'}{K}} = 1\pm\frac{2q^m\sin x}{(1+q^{2m})^2} = \frac{1\pm 2q^m\sin x+q^{2m}}{1+q^{2m}}$$

$$\frac{-\cos\frac{mi\pi K'}{K}}{\sin^2\frac{mi\pi K'}{K}-y^2} = \frac{2q^m(1+q^{2m})}{1-2q^{2m}\cos 2x+q^{4m}}$$
$$\frac{i\sin\frac{mi\pi K'}{K}}{\sin^2\frac{mi\pi K'}{K}-y^2} = \frac{2q^m(1-q^{2m})}{1-2q^{2m}\cos 2x+q^{4m}}$$

Having prepared these things and having for the sake of brevity put :

$$A = \left\{\frac{(1-q)}{(1-q^2)}\frac{(1-q^3)}{(1-q^4)}\frac{(1-q^5)\cdots}{(1-q^6)\cdots}\right\}^2$$
$$B = \left\{\frac{(1-q)}{(1+q^2)}\frac{(1-q^3)}{(1+q^4)}\frac{(1-q^5)\cdots}{(1+q^6)\cdots}\right\}^2$$
$$C = \left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^3)}\frac{(1-q^5)\cdots}{(1+q^5)\cdots}\right\}^2,$$

the following fundamental expansions of the elliptic functions into infinite products result:

$$(1.)\quad \sin\mathrm{am}\frac{2Kx}{\pi} = \frac{2AK}{\pi}\sin x\cdot\frac{(1-2q^2\cos 2x+q^4)}{(1-2q\cos 2x+q^2)}\frac{(1-2q^4\cos 2x+q^8)}{(1-2q^3\cos 2x+q^6)}\frac{(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q^5\cos 2x+q^{10})\cdots}$$
$$(2.)\quad \sin\mathrm{am}\frac{2Kx}{\pi} = B\cos x\cdot\frac{(1+2q^2\cos 2x+q^4)}{(1-2q\cos 2x+q^2)}\frac{(1+2q^4\cos 2x+q^8)}{(1-2q^3\cos 2x+q^6)}\frac{(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q^5\cos 2x+q^{10})\cdots}$$
$$(3.)\quad \Delta\,\mathrm{am}\frac{2Kx}{\pi} = C\cdot\frac{(1+2q\cos 2x+q^2)}{(1-2q\cos 2x+q^2)}\frac{(1+2q^3\cos 2x+q^6)}{(1-2q^3\cos 2x+q^6)}\frac{(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q^5\cos 2x+q^{10})\cdots}$$

$$(4.)\quad \sqrt{\frac{1-\sin\mathrm{am}\frac{2Kx}{\pi}}{1+\sin\mathrm{am}\frac{2Kx}{\pi}}} = \sqrt{\frac{1-\sin x}{1+\sin x}}\cdot\frac{(1-2q\sin x+q^2)}{(1+2q\sin x+q^2)}\frac{(1-2q^2\sin x+q^4)}{(1+2q^2\sin x+q^4)}\frac{(1-2q^3\sin x+q^6)\cdots}{(1+2q^3\sin x+q^6)\cdots}$$
$$(5.)\quad \sqrt{\frac{1-k\sin\mathrm{am}\frac{2Kx}{\pi}}{1+k\sin\mathrm{am}\frac{2Kx}{\pi}}} = \sqrt{\frac{1-\sin x}{1+\sin x}}\cdot\frac{(1-2\sqrt{q}\sin x+q)}{(1+2\sqrt{q}\sin x+q)}\frac{(1-2\sqrt{q^3}\sin x+q^3)}{(1+2\sqrt{q^3}\sin x+q^3)}\frac{(1-2\sqrt{q^5}\sin x+q^5)\cdots}{(1+2\sqrt{q^5}\sin x+q^5)\cdots}$$

and another system of formulas which the resolution into simple fractions yields:

$$(6.)\quad \sin\operatorname{am}\frac{2Kx}{\pi} = \frac{2\pi}{kK}\sin x\left(\frac{\sqrt{q}(1+q)}{1-2q\cos 2x+q^2}+\frac{\sqrt{q^3}(1+q^3)}{1-2q^3\cos 2x+q^6}+\frac{\sqrt{q^5}(1+q^5)}{1-2q^5\cos 2x+q^{10}}+\cdots\right)$$

$$(7.)\quad \cos\operatorname{am}\frac{2Kx}{\pi} = \frac{2\pi}{kK}\cos x\left(\frac{\sqrt{q}(1-q)}{1-2q\cos 2x+q^2}-\frac{\sqrt{q^3}(1-q^3)}{1-2q^3\cos 2x+q^6}+\frac{\sqrt{q^5}(1-q^5)}{1-2q^5\cos 2x+q^{10}}-\cdots\right)$$

To these we add the following from the same source:

$$(8.)\quad 1-\Delta\operatorname{am}\frac{2Kx}{\pi} = \frac{4\pi\sin^2 x}{K}\left(\frac{q\left(\frac{1+q}{1-q}\right)}{1-2q\cos 2x+q^2}-\frac{q^3\left(\frac{1+q^3}{1-q^3}\right)}{1-2q^3\cos 2x+q^6}+\frac{q^5\left(\frac{1+q^5}{1-q^5}\right)}{1-2q^5\cos 2x+q^{10}}-\cdots\right)$$

$$(9.)\quad \operatorname{am}\frac{2Kx}{\pi} = \pm x+2\arctan\frac{(1+q)\tan x}{1-q}-2\arctan\frac{(1+q^3)\tan x}{1-q^3}+2\arctan\frac{(1+q^5)\tan x}{1-q^5}-\cdots$$

In the last formula the upper sign is to be chosen, if one stops at a negative term, the upper if one stops at a positive term.

36.

Let us consider the formulas (1.), (2.), (3.) in which especially the values of the quantities we denoted by A, B, C are to be found. They are easily found from formulas (3.), (1.) by putting $x = \frac{\pi}{2}$:

$$k' = C\left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^3)}\frac{(1-q^5)\cdots}{(1+q^5)\cdots}\right\}^2 = CC,$$

whence it follows

$$1 = \frac{2AK}{\pi}\left\{\frac{(1+q^2)}{(1+q)}\frac{(1+q^4)}{(1+q^3)}\frac{(1+q^6)\cdots}{(1+q^5)\cdots}\right\}^2 = \frac{2AK}{\pi}\cdot\frac{C}{B} = \frac{2\sqrt{k'}AK}{\pi B},$$

whence it is

$$B = \frac{2\sqrt{k'}AK}{\pi}.$$

But to find the value of A other artifices are to be used.

Let us put $e^{ix} = U$: If x is changed into $x + \frac{i\pi K'}{2K}$, U goes over into $\sqrt{q}U$, $\sin\operatorname{am}\frac{2Kx}{\pi}$ into

$$\sin\operatorname{am}\left(\frac{2Kx}{\pi} + iK'\right) = \frac{1}{k\sin\operatorname{am}\frac{2Kx}{\pi}}.$$

But from formula (1.) we obtain:

$$\sin\operatorname{am}\frac{2Kx}{\pi} = \frac{AK}{\pi}\left(\frac{U-U^{-1}}{i}\right)\frac{[(1-q^2U^2)(1-q^4U^2)\cdots]}{[(1-qU^2)(1-q^3U^2)\cdots]}\cdots\frac{[(1-q^2U^{-2})(1-q^4U^{-2})\cdots]}{[(1-qU^{-2})(1-q^3U^{-2})\cdots]},$$

whence by changing x into $x + \frac{i\pi K'}{2K}$:

$$\frac{1}{k\sin\operatorname{am}\frac{2Kx}{\pi}} = \frac{AK}{\pi}\left(\frac{\sqrt{q}U - \sqrt{q^{-1}}U^{-1}}{i}\right)\frac{[(1-q^3U^2)(1-q^5U^2)\cdots]}{[(1-^2U^2)(1-q^4U^2)\cdots]}\cdots\frac{[(1-q^2U^{-2})(1-U^{-2})\cdots]}{[(1-q^3U^{-2})(1-q^2U^{-2})\cdots]},$$

having multiplied those by each other, because it is:

$$\frac{\sqrt{q}U - \sqrt{q^{-1}}U^{-1}}{1-U^{-2}} = -\frac{1}{\sqrt{q}}\cdot\frac{1-qU^2}{U-U^{-1}},$$

it results:

$$\frac{1}{k} = \frac{1}{\sqrt{q}}\left(\frac{AK}{\pi}\right)^2 \quad \text{or} \quad A = \frac{\pi\sqrt[4]{q}}{\sqrt{k}K}; \quad \text{hence} \quad \frac{2KA}{\pi} = \frac{2\sqrt[4]{q}}{\sqrt{k}}.$$

Hence it will be $B = \frac{2\sqrt{k'}AK}{\pi} = 2\sqrt[4]{q}\sqrt{\frac{k'}{k}}$. Therefore, it is:

$$\begin{aligned}
\sin\operatorname{am}\frac{2Kx}{\pi} &= \frac{1}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}\\
\cos\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{\frac{k'}{k}}\cdot\frac{2\sqrt[4]{q}\cos x(1+2q^2\cos 2x+q^4)(1+2q^4\cos 2x+q^8)(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}\\
\Delta\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{k'}\cdot\frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}.
\end{aligned}$$

Having multiplied these equations by each other:

$$\begin{aligned}
B &= 2\sqrt[4]{q}\sqrt{\frac{k'}{k}} = \left\{\frac{(1-q)}{(1+q^2)}\frac{(1-q^3)}{(1+q^4)}\frac{(1-q^5)\cdots}{(1+q^6)\cdots}\right\}^2\\
C &= \sqrt{k'} = \left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^2)}\frac{(1-q^5)\cdots}{(1+q^3)\cdots}\right\}^2,
\end{aligned}$$

it results:

$$\frac{2\sqrt[4]{q}k'}{\sqrt{k}} = \frac{[(1-q)(1-q^3)(1-q^5)\cdots]^4}{(1+q)(1+q^2)(1+q^3)\cdots]^2}.$$

But according to Euler in *Introductio (de Partitione Numerorum)* it is:

$$\begin{aligned}(1+q)(1+q^2)(1+q^3)\cdots &= \frac{(1-q^2)(1-q^4)(1-q^6)\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} \\ &= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots},\end{aligned}$$

whence we obtain:

$$(1.) \qquad [(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots]^6 = \frac{2\sqrt[4]{q}k'}{\sqrt{k}}.$$

Recalling the formula:

$$A = \frac{\pi\sqrt[4]{q}}{\sqrt{k}K} = \left\{\frac{(1-q)(1-q^3)(1-q^5)\cdots}{(1-q^2)(1-q^4)(1-q^6)\cdots}\right\}^2,$$

it is:

$$(2.) \qquad [(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots]^6 = \frac{2kk'K^3}{\pi^3\sqrt{q}},$$

whence also:

$$(3.) \qquad [(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots]^6 = \frac{4\sqrt{k}k'k'K^3}{\pi^3\sqrt[4]{q}}.$$

One can add these formulas which easily follow:

$$\begin{aligned}
&(4.) \qquad [(1+q)(1+q^3)(1+q^5)(1+q^7)\cdots]^6 &&= \frac{2\sqrt[4]{q}}{\sqrt{kk'}} \\
&(5.) \qquad [(1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots]^6 &&= \frac{k}{4\sqrt{k'}\sqrt{q}} \\
&(6.) \qquad [(1+q)(1+q^2)(1+q^3)(1+q^4)\cdots]^6 &&= \frac{\sqrt{k}}{2k'\sqrt[4]{q}}.
\end{aligned}$$

From these one also concludes:

$$\begin{array}{lll}
(7.) & k & = 4\sqrt{q}\left\{\frac{(1+q^2)}{(1+q)}\frac{(1+q^4)}{(1+q^3)}\frac{(1+q^6)\cdots}{(1+q^5)\cdots}\right\}^4 \\
(8.) & k' & = \left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^3)}\frac{(1-q^5)\cdots}{(1+q^5)\cdots}\right\}^4 \\
(9.) & \frac{2K}{\pi} & = \left\{\frac{(1-q^2)}{(1-q)}\frac{(1-q^4)}{(1-q^3)}\frac{(1-q^6)\cdots}{(1-q^5)\cdots}\right\}^2 \left\{\frac{(1+q)}{(1+q^2)}\frac{(1+q^3)}{(1+q^4)}\frac{(1+q^5)\cdots}{(1+q^6)\cdots}\right\}^2 \\
(10.) & \frac{2kK}{\pi} & = 4\sqrt{q}\left\{\frac{(1-q^4)}{(1-q^2)}\frac{(1-q^8)}{(1-q^6)}\frac{(1-q^{12})\cdots}{(1-q^{10})\cdots}\right\}^2 \\
(11.) & \frac{2k'K}{\pi} & = \left\{\frac{(1-q)}{(1+q)}\frac{(1-q^2)}{(1+q^2)}\frac{(1-q^3)\cdots}{(1+q^3)\cdots}\right\}^2 \\
(12.) & \frac{2\sqrt{k}K}{\pi} & = 2\sqrt[4]{q}\left\{\frac{(1-q^2)}{(1-q)}\frac{(1-q^4)}{(1-q^3)}\frac{(1-q^6)\cdots}{(1-q^5)\cdots}\right\}^2 \\
(13.) & \frac{2\sqrt{k'}K}{\pi} & = \left\{\frac{(1-q^2)}{(1+q^2)}\frac{(1-q^4)}{(1+q^4)}\frac{(1-q^6)\cdots}{(1+q^6)\cdots}\right\}^2.
\end{array}$$

From formulas (7.), (8.) this non-obvious identity follows:

$$(14.)\ [(1-q)(1-q^3)(1-q^5)\cdots]^8 + 16q[(1+q^2)(1+q^4)(1+q^6)\cdots]^8 = [(1+q)(1+q^3)(1+q^5)\cdots]^8.$$

37.

We have seen above, where the properties of the modular equations were discussed, having changed k into to $\frac{1}{k}$ that k goes over into $k(K-iK')$, K' into kK'; further that it is:

$$\begin{aligned}
\sin\mathrm{am}\left(ku, \frac{ik'}{k}\right) &= \cos\mathrm{coam}(u, k') \\
\cos\mathrm{am}\left(ku, \frac{ik'}{k}\right) &= \sin\mathrm{coam}(u, k') \\
\Delta\,\mathrm{am}\left(ku, \frac{ik'}{k}\right) &= \frac{1}{\Delta\,\mathrm{am}(u, k')}.
\end{aligned}$$

Having interchanged k and k' from this it follows, if k' goes over into $\frac{1}{k'}$ or k into $\frac{ik}{k'}$, that at the same time K goes over into $k'K$, K' into $k'(K' - iK)$; further that it is:

$$\begin{aligned}
\sin\operatorname{am}\left(k'u, \frac{ik}{k'}\right) &= \cos\operatorname{coam} u \\
\cos\operatorname{am}\left(k'u, \frac{ik}{k'}\right) &= \sin\operatorname{coam} u \\
\Delta\operatorname{am}\left(k'u, \frac{ik}{k'}\right) &= \frac{1}{\Delta\operatorname{am} u},
\end{aligned}$$

whence also:

$$\operatorname{am}\left(k'u, \frac{ik}{k'}\right) = \frac{\pi}{2} - \operatorname{coam} u.$$

But having changed K into $k'K$, K' into $k'(K' - iK)$, $q = e^{\frac{\pi K'}{K}}$ goes over into $-q$, whence it vice versa follows:

Theorem I

Having changed q into $-q$ we have:

$$\begin{array}{llllll}
k & \text{goes over into} & \frac{ik}{k'}, & k' & \text{goes over into} & \frac{1}{k'} \\
K & \text{goes over into} & k'K, & K' & \text{goes over into} & k'(K' - iK)
\end{array}$$

$$\begin{array}{lll}
\sin\operatorname{am}\frac{2Kx}{\pi} & \text{goes over into} & \cos\operatorname{coam}\frac{2Kx}{\pi} \\
\cos\operatorname{am}\frac{2Kx}{\pi} & \text{goes over into} & \sin\operatorname{coam}\frac{2Kx}{\pi} \\
\Delta\operatorname{am}\frac{2Kx}{\pi} & \text{goes over into} & \frac{1}{\Delta\operatorname{am}\frac{2Kx}{\pi}} \\
\operatorname{am}\frac{2Kx}{\pi} & \text{goes over into} & \frac{\pi}{2} - \operatorname{coam}\frac{2Kx}{\pi};
\end{array}$$

having changed q to $-q$ and x to $x - \frac{\pi}{2}$

$$\begin{array}{lll}
\operatorname{am} \dfrac{2Kx}{\pi} & \text{goes over into} & \dfrac{\pi}{2} - \operatorname{am} \dfrac{2Kx}{\pi} \\
\sin \operatorname{am} \dfrac{2Kx}{\pi} & \text{goes over into} & \cos \operatorname{am} \dfrac{2Kx}{\pi} \\
\cos \operatorname{am} \dfrac{2Kx}{\pi} & \text{goes over into} & \sin \operatorname{am} \dfrac{2Kx}{\pi} \\
\Delta \operatorname{am} \dfrac{2Kx}{\pi} & \text{goes over into} & \dfrac{1}{k'} \Delta \operatorname{am} \dfrac{2Kx}{\pi}.
\end{array}$$

At last, let us investigate how the elliptic functions transform having changed q either to q^2 or to $\sqrt{q}$.

We saw above that the modulus λ derived from the modulus k by means of the first real transformation of n-th order enjoys the extraordinary property that it is:

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K};$$

therefore, having changed k into λ, $q = e^{\frac{-\pi K'}{K}}$ goes over into q^n. The same, proved by us on the transformations of odd order in general, was proved already by Legendre on the transformation of second order long time ago, of course, having put $\lambda = \frac{1-k'}{1+k'}$ he proved that it is:

$$\Lambda = \frac{1+k'}{2}K, \quad \Lambda' = (1+k')K', \quad \frac{\Lambda'}{\Lambda} = 2\frac{K'}{K},$$

whence we see, having changed k into $\frac{1-k'}{1+k'}$, that q goes over into q^2. Hence we vice versa obtain

Theorem II.

Having changed q into q^2, k goes over into $\frac{1-k'}{1+k'}$, K into $\frac{1+k'}{2}K$, whence also:

$$
\begin{array}{llll}
k' & \text{goes over into } \dfrac{2\sqrt{k'}}{1+k'} & 1+k & \text{goes over into } \dfrac{2}{1+k'} \\
k'K & \text{goes over into } \sqrt{k'}K & 1-k & \text{goes over into } \dfrac{2k'}{1+k'} \\
\sqrt{k} & \text{goes over into } \dfrac{k}{1+k'} & 1+k' & \text{goes over into } \dfrac{(1+\sqrt{k})^2}{1+k'} \\
\sqrt{k}K & \text{goes over into } \dfrac{kK}{2} & 1-k' & \text{goes over into } \dfrac{(1+\sqrt{k'})^2}{1+k'}
\end{array}
$$

From the inversion of this theorem one obtains another

Theorem III.

Having changed q into $\sqrt{q}$, k goes over into $\frac{2\sqrt{k}}{1+k}$, K into $(1+k)K$, whence also:

$$
\begin{array}{llll}
k' & \text{goes over into } \dfrac{1-k}{1+k} & 1+k & \text{goes over into } \dfrac{(1+\sqrt{k})^2}{1+k} \\
\sqrt{k'} & \text{goes over into } \dfrac{k'}{1+k} & 1-k & \text{goes over into } \dfrac{(1-\sqrt{k})^2}{1+k} \\
kK & \text{goes over into } 2\sqrt{k}K & 1+k' & \text{goes over into } \dfrac{2}{1+k} \\
\sqrt{k}K & \text{goes over into } k'K & 1-k' & \text{goes over into } \dfrac{2k}{1+k}
\end{array}
$$

These three theorems, by means of which either it is possible to derive even more formulas from others or confirm formulas found from other sources, are confirmed by the expansions propounded in § 35 and § 36 in many ways and they will have a very frequent use in the following.

38.

We want to denote the quantities into which, after having put q^r instead of q, k, k', K go over, by $k^{(r)}$, $k^{(r)\prime}$, $K^{(r)}$ so that $k^{(r)}$ is the modulus found by the first real transformation of r-th order and $k^{(r)\prime}$ its complement. Let us in the equation:

$$\sqrt{k'}\left\{\frac{(1-q)}{(1+q)}\frac{(1-q^3)}{(1+q^3)}\frac{(1-q^5)}{(1+q^5)}\frac{(1-q^7)\cdots}{(1+q^7)\cdots}\right\}^2$$

instead of q successively put q^2, q^4, q^8, q^{16} etc. Multiplying all equations it results:

$$\sqrt{k^{(2)\prime}k^{(4)\prime}k^{(8)\prime}k^{(16)\prime}\cdots} = \left\{\frac{(1-q^2)}{(1+q^2)}\frac{(1-q^4)}{(1+q^4)}\frac{(1-q^6)}{(1+q^6)}\frac{(1-q^8)\cdots}{(1+q^8)\cdots}\right\}^2;$$

but we found:

$$\left\{\frac{(1-q^2)}{(1+q^2)}\frac{(1-q^4)}{(1+q^4)}\frac{(1-q^6)}{(1+q^6)}\frac{(1-q^8)\cdots}{(1+q^8)\cdots}\right\}^2 = \frac{2\sqrt{k'}K}{\pi},$$

whence:

$$(1.)\qquad \frac{2K}{\pi} = \sqrt{\frac{k^{(2)\prime}k^{(4)\prime}k^{(8)\prime}k^{(16)\prime}\cdots}{k'}}.$$

Because it is $k^{(2)\prime} = \frac{2\sqrt{k'}}{1+k'}$, from $(1.)$ it is:

$$\left(\frac{2K}{\pi}\right)^2 = \frac{1}{k'}\cdot\frac{2\sqrt{k'}}{1+k'}\cdot\frac{2\sqrt{k^{(2)\prime}}}{1+k^{(2)\prime}}\cdot\frac{2\sqrt{k^{(4)\prime}}}{1+k^{(4)\prime}}\cdot\frac{2\sqrt{k^{(8)\prime}}}{1+k^{(8)\prime}}\cdots,$$

whence having divided by $(1.)$ we have:

$$(2.)\qquad \frac{2K}{\pi} = \frac{2}{1+k'}\cdot\frac{2}{1+k^{(2)\prime}}\cdot\frac{2}{1+k^{(4)\prime}}\cdot\frac{2}{1+k^{(8)\prime}}\cdots$$

This formula is also obtained because the following hold:

$$\begin{aligned}\frac{2K}{\pi} &= \frac{2K^{(2)}}{\pi}\cdot\frac{2}{1+k'}\\ \frac{2K^{(2)}}{\pi} &= \frac{2K^{(4)}}{\pi}\cdot\frac{2}{1+k^{(2)\prime}}\\ \frac{2K^{(4)}}{\pi} &= \frac{2K^{(8)}}{\pi}\cdot\frac{2}{1+k^{(4)\prime}}\\ &\cdots,\end{aligned}$$

whence, because as r increases to infinity the limit of the expression $\frac{2K^{(r)}}{\pi}$ is 1, having expanded the infinite product, (2.) results. Having put:

$$\begin{aligned} m &= 1, & n &= k' \\ m' &= \frac{m+n}{2}, & n' &= \sqrt{nm} \\ m'' &= \frac{m'+n'}{2}, & n'' &= \sqrt{n'm'} \\ m''' &= \frac{m''+n''}{2}, & n''' &= \sqrt{n''m''} \end{aligned}$$

it is:

$$\begin{aligned} k^{(2)\prime} &= \frac{2\sqrt{k'}}{1+k'} &= \frac{n'}{m'} \\ k^{(4)\prime} &= \frac{2\sqrt{(2)k'}}{1+k^{(2)\prime}} &= \frac{n''}{m''} \\ k^{(8)\prime} &= \frac{2\sqrt{(4)k'}}{1+k^{(4)\prime}} &= \frac{n'''}{m'''} \\ &\cdots, \end{aligned}$$

whence:

$$\frac{2}{1+k'} = \frac{m}{m'}, \quad \frac{2}{1+k^{(2)\prime}} = \frac{m'}{m''}, \quad \frac{2}{1+k^{(4)\prime}} = \frac{m''}{m'''}, \ldots$$

and hence:

$$\frac{2K}{\pi} = \frac{m}{m'} \cdot \frac{m'}{m''} \cdot \frac{m''}{m'''} \cdot \frac{m'''}{m''''} \cdots$$

or while μ denotes the common limit to which $m^{(p)}$, $n^{(p)}$ converge as p grows to infinity:

$$(3.) \qquad \frac{2K}{\pi} = \frac{1}{\mu},$$

These results are known from other sources.

Let us now again in formula:

$$\Delta\,\mathrm{am}\,\frac{2Kx}{\pi} = \sqrt{k'}\frac{(1+2q\cos 2x+q^2)}{(1-2q\cos 2x+q^2)}\cdot\frac{(1+2q^3\cos 2x+q^6)}{(1-2q^3\cos 2x+q^6)}\cdot\frac{(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q^5\cos 2x+q^{10})\cdots}$$

instead of q successively put q^2, q^4, q^8, etc.; further let:

$$S = \Delta\,\mathrm{am}\left(\frac{2K^{(2)}x}{\pi}, k^{(2)}\right)\Delta\,\mathrm{am}\left(\frac{2K^{(4)}x}{\pi}, k^{(4)}\right)\Delta\,\mathrm{am}\left(\frac{2K^{(8)}x}{\pi}, k^{(8)}\right)\cdots$$

Because having constructed the infinite product it is:

$$\frac{2\sqrt{k'}K}{\pi} = \sqrt{k^{(2)\prime}k^{(4)\prime}k^{(8)\prime}k^{(16)\prime}\cdots},$$

we obtain:

$$S = \frac{2\sqrt{k'}K}{\pi}\cdot\frac{(1+2q^2\cos 2x+q^4)}{(1-2q^2\cos 2x+q^4)}\cdot\frac{(1+2q^4\cos 2x+q^8)}{(1-2q^4\cos 2x+q^8)}\cdot\frac{(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q^6\cos 2x+q^{12})\cdots}$$

But from the formulas:

$$\sin\mathrm{am}\,\frac{2Kx}{\pi} = \frac{2}{\sqrt{k}}\cdot\frac{\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}$$

$$\cos\mathrm{am}\,\frac{2Kx}{\pi} = 2\sqrt{\frac{k'}{k}}\cdot\frac{\sqrt[4]{q}\cos x(1+2q^2\cos 2x+q^4)(1+2q^4\cos 2x+q^8)(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}$$

we obtain:

$$\Delta\,\mathrm{am}\,\frac{2Kx}{\pi} = \frac{1}{\sqrt{k'}}\cdot\frac{\tan x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{(1+2q\cos 2x+q^4)(1+2q^3\cos 2x+q^8)(1+2q^6\cos 2x+q^{12})\cdots}$$

whence this memorable formula results:

$$(4.)\qquad \tan x = \frac{S\cdot\tan\mathrm{am}\,\frac{2Kx}{\pi}}{\frac{2K}{\pi}}.$$

To demonstrate the same by means of known formulas let us recall a formula for the transformations of second order which Gauss exhibited in the treatise: *"Determinatio Attractionis"* etc.:

$$\sin \operatorname{am} \frac{2Kx}{\pi} = \frac{(1+k^{(2)}) \sin \operatorname{am} \left(\frac{2K^{(2)}x}{\pi}, k^{(2)}\right)}{1+k^{(2)} \sin \operatorname{am} \left(\frac{2K^{(2)}x}{\pi}, k^{(2)}\right)},$$

which having for the sake of brevity put:

$$\operatorname{am} \left(\frac{2K^{(r)}x}{\pi}, k^{(r)}\right) = \varphi^{(r)}, \quad \Delta \operatorname{am} \left(\frac{2K^{(r)}x}{\pi}, k^{(r)}\right) = \Delta^{(r)},$$

it is exhibited as this:

$$\sin \varphi = \frac{(1+k^{(2)}) \sin \varphi^{(2)}}{1+k^{(2)} \sin \varphi^{(2)}},$$

whence also:

$$\begin{aligned}
\cos \varphi &= \frac{\cos \varphi^{(2)} \Delta^{(2)}}{1+k^{(2)} \sin^2 \varphi^{(2)}} \\
\Delta\varphi &= \frac{1-k^{(2)} \sin^2 \varphi^{(2)}}{1+k^{(2)} \sin^2 \varphi^{(2)}} \\
\tan \varphi &= \frac{(1+k^{(2)}) \tan \varphi^{(2)}}{\Delta^{(2)}}
\end{aligned}$$

The last formula can also be represented this way:

$$\frac{\tan \varphi}{\frac{2K}{\pi}} = \frac{\tan \varphi^{(2)}}{\frac{2K^{(2)}}{\pi}} \cdot \frac{1}{\Delta^{(2)}},$$

whence having successively put $q^2, q^4, q^8, \cdots$ instead of q, having done which k, K, φ go over into $k^{(2)}, k^{(4)}, k^{(8)}, \cdots; K^{(2)}, K^{(4)}, K^{(8)}, \cdots; \varphi^{(2)}, \varphi^{(4)}, \varphi^{(8)}, \cdots,$ we obtain:

$$\frac{\tan\varphi^{(2)}}{\frac{2K^{(2)}}{\pi}} = \frac{\tan\varphi^{(4)}}{\frac{2K^{(4)}}{\pi}} \cdot \frac{1}{\Delta^{(4)}}$$

$$\frac{\tan\varphi^{(4)}}{\frac{2K^{(4)}}{\pi}} = \frac{\tan\varphi^{(8)}}{\frac{2K^{(8)}}{\pi}} \cdot \frac{1}{\Delta^{(8)}}$$

$$\frac{\tan\varphi^{(8)}}{\frac{2K^{(8)}}{\pi}} = \frac{\tan\varphi^{(16)}}{\frac{2K^{(16)}}{\pi}} \cdot \frac{1}{\Delta^{(16)}}$$

$$\cdots$$

Now, the limit of the expression

$$\frac{\tan\varphi^{(p)}}{\frac{2K^{(p)}}{\pi}} = \frac{\tan\operatorname{am}\left(\frac{2K^{(p)}x}{\pi}, k^{(p)}\right)}{\frac{2K^{(p)}}{\pi}},$$

as p grows to infinity, is:

$$\tan x;$$

for, then it is $k^{(p)} = 0$, $K^{(p)} = \frac{\pi}{2}$, $\operatorname{am}(u, k^{(p)}) = u$; hence having taken the infinite product and having, as above, put $S = \Delta^{(2)}\Delta^{(4)}\Delta^{(8)}\cdots$, it results:

$$\frac{\tan\varphi}{\frac{2k}{\pi}} = \frac{\tan x}{S},$$

which is the formula to be demonstrated.

From the formula:

$$\tan x = \frac{S \cdot \tan\varphi}{\frac{2K}{\pi}}$$

an elegant algorithm for the computation of *indefinite* elliptic integrals of the first kind can be derived; and this by means of the easy to prove formula:

$$\Delta^{(2)} = \sqrt{\frac{2(\Delta + k')}{(1 + k')(1 + \Delta)}}$$

For this aim, we state the following

Theorem

Having put:

$$\int_0^\varphi \frac{d\varphi}{\sqrt{mm\cos^2\varphi + nn\sin^2\varphi}} = \Phi$$
$$\sqrt{mm\cos^2\varphi + nn\sin^2\varphi} = \Delta,$$

form the expressions:

$$\begin{array}{llllll}
\dfrac{m+n}{2} & = m' & \sqrt{mn} & = n' & \Delta' & = \sqrt{\dfrac{mm'(\Delta+n)}{m+\Delta}} \\
\dfrac{m'+n'}{2} & = m'' & \sqrt{m'n'} & = n'' & \Delta'' & = \sqrt{\dfrac{m'm''(\Delta'+n')}{m'+\Delta'}} \\
\dfrac{m''+n''}{2} & = m''' & \sqrt{m''n''} & = n''' & \Delta''' & = \sqrt{\dfrac{m''m'''(\Delta''+n'')}{m''+\Delta''}} \\
& \cdots & & \cdots & & \cdots;
\end{array}$$

μ denoting the common limit to which the quantities $m^{(p)}$, $\Delta^{(p)}$, $n^{(p)}$ as p increases very rapidly converge, it will be:

$$\tan\mu\Phi = \frac{\Delta'\Delta''\Delta'''\cdots}{mm'm''\cdots}\cdot\tan\varphi.$$

By the same methods we used in the preceding one also finds the value of the infinite product:

$$\frac{2\sqrt[4]{q}}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q^2}}{\sqrt{k^{(2)}}}\cdot\frac{2\sqrt[4]{q^4}}{\sqrt{k^{(4)}}}\cdot\frac{2\sqrt[4]{q^8}}{\sqrt{k^{(8)}}}\cdots$$

For this aim, recall the formulas from § 36 (4.), (5.):

$$[(1+q)(1+q^3)(1+q^5)(1+q^7)\cdots]^6 = \frac{2\sqrt[4]{q}}{\sqrt{kk'}}$$
$$[(1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots]^6 = \frac{k}{4\sqrt{k'}\sqrt{q}},$$

the latter of which results from the first having successively put q^2, q^4, q^8 etc. instead of q and taken the infinite product, whence we obtain:

$$\frac{k}{4\sqrt{k'}\sqrt{q}} = \frac{2\sqrt[4]{q^2}}{\sqrt{k^{(2)}k^{(2)\prime}}} \cdot \frac{2\sqrt[4]{q^4}}{\sqrt{k^{(4)}k^{(4)\prime}}} \cdot \frac{2\sqrt[4]{q^8}}{\sqrt{k^{(8)}k^{(8)\prime}}} \cdots$$

But we already found (1.):

$$\frac{2K}{\pi} = \sqrt{\frac{k^{(2)\prime}k^{(4)\prime}k^{(8)\prime}\cdots}{k'}},$$

whence

$$(5.) \qquad \frac{\sqrt{k}}{2\sqrt[4]{q}} \cdot \frac{2K}{\pi} = \frac{2\sqrt[4]{q}}{\sqrt{k}} \cdot \frac{2\sqrt[4]{q^2}}{\sqrt{k^{(2)}}} \cdot \frac{2\sqrt[4]{q^4}}{\sqrt{k^{(4)}}} \cdot \frac{2\sqrt[4]{q^8}}{\sqrt{k^{(8)}}} \cdots$$

These things might seem not to be connected to our actual subject; but because they are elegant and very helpful to understand the nature of the propounded expansion, it is useful to have explained them.

2.1 Expansion of elliptic Functions into series of Sines or Cosines of multiples of the Argument

39.

From the formulas given above:

$$(1.)\ \sin\operatorname{am}\frac{2Kx}{\pi} = \frac{2\sqrt[4]{q}}{\sqrt{k}} \sin x \cdot \frac{(1-2q^2\cos 2x+q^4)\,(1-2q^4\cos 2x+q^8)\,(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)\,(1-2q^3\cos 2x+q^6)\,(1-2q^5\cos 2x+q^{10})\cdots}$$

$$(2.)\ \cos\operatorname{am}\frac{2Kx}{\pi} = \frac{2\sqrt[4]{q}\sqrt{k'}}{\sqrt{k}} \cos x \cdot \frac{(1+2q^2\cos 2x+q^4)\,(1+2q^4\cos 2x+q^8)\,(1+2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)\,(1-2q^3\cos 2x+q^6)\,(1-2q^5\cos 2x+q^{10})\cdots}$$

$$(3.)\ \Delta\operatorname{am}\frac{2Kx}{\pi} = \sqrt{k'} \cdot \frac{(1+2q\cos 2x+q^2)\,(1+2q^3\cos 2x+q^6)\,(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q\cos 2x+q^2)\,(1-2q^3\cos 2x+q^6)\,(1-2q^5\cos 2x+q^{10})\cdots}$$

And

$$(4.)\ \sqrt{\frac{1-\sin\operatorname{am}\frac{2Kx}{\pi}}{1+\sin\operatorname{am}\frac{2Kx}{\pi}}} = \sqrt{\frac{1-\sin x}{1+\sin x}} \cdot \frac{(1-2q\sin x+q^2)\,(1-2q^2\sin x+q^4)\,(1-2q^3\sin x+q^6)\cdots}{(1+2q\sin x+q^2)\,(1+2q^2\sin x+q^4)\,(1+2q^3\sin x+q^6)\cdots}$$

$$(5.)\ \sqrt{\frac{1-k\sin\operatorname{am}\frac{2Kx}{\pi}}{1+k\sin\operatorname{am}\frac{2Kx}{\pi}}} = \frac{(1-2\sqrt{q}\sin x+q)\,(1-2\sqrt{q^3}\sin x+q^3)\,(1-2\sqrt{q^5}\sin x+q^5)\cdots}{(1+2\sqrt{q}\sin x+q)\,(1+2\sqrt{q^3}\sin x+q^3)\,(1+2\sqrt{q^5}\sin x+q^5)\cdots},$$

having expanded the logarithms of the single products on the one side of the equations, after some obvious reductions, these follow:

$$\begin{aligned}
&(6.)\quad \ln\sin\operatorname{am}\frac{2Kx}{\pi} = \ln\left\{\frac{2\sqrt[4]{q}}{\sqrt{k}}\sin x\right\} + \frac{2q\cos 2x}{1+q} + \frac{2q^2\cos 4x}{2(1+q^2)} + \frac{2q^3\cos 6x}{3(1+q^3)} + \cdots\\
&(7.)\quad \ln\cos\operatorname{am}\frac{2Kx}{\pi} = \ln\left\{2\sqrt[4]{q}\sqrt{\frac{k'}{k}}\cos x\right\} + \frac{2q\cos 2x}{1-q} + \frac{2q^2\cos 4x}{2(1+q^2)} + \frac{2q^3\cos 6x}{3(1-q^3)} + \cdots\\
&(8.)\quad \ln\Delta\operatorname{am}\frac{2Kx}{\pi} = \ln\sqrt{k'} + \frac{4q\cos 2x}{1-q^2} + \frac{4q^3\cos 4x}{3(1-q^6)} + \frac{4q^5\cos 10x}{5(1-q^{10})} + \cdots
\end{aligned}$$

and

$$\begin{aligned}
&(9.)\quad \ln\sqrt{\frac{1+\sin\operatorname{am}\frac{2Kx}{\pi}}{1+-\sin\operatorname{am}\frac{2Kx}{\pi}}} = \ln\sqrt{\frac{1+\sin x}{1-\sin x}} + \frac{4q\sin x}{1-q} - \frac{4q^3\sin 3x}{3(1-q^3)} + \frac{4q^5\sin 5x}{5(1-q^5)} - \cdots\\
&(10.)\quad \ln\sqrt{\frac{1+k\sin\operatorname{am}\frac{2Kx}{\pi}}{1-k\sin\operatorname{am}\frac{2Kx}{\pi}}} = \frac{4\sqrt{q}\sin x}{1-q} - \frac{4\sqrt{q^3}\sin 3x}{3(1-q^3)} + \frac{4\sqrt{q^5}\sin 5x}{5(1-q^5)} - \cdots
\end{aligned}$$

Having differentiated these formulas, if we note the following easy to prove differential formulas:

$$\begin{aligned}
\frac{d\ln\sin\operatorname{am}\frac{2Kx}{\pi}}{dx} &= \frac{2k'K}{\pi}\cdot\frac{\cos\operatorname{am}\frac{2Kx}{\pi}}{\cos\operatorname{coam}\frac{2Kx}{\pi}}\\
-\frac{d\ln\sin\operatorname{am}\frac{2Kx}{\pi}}{dx} &= \frac{2K}{\pi}\cdot\frac{\sin\operatorname{am}\frac{2Kx}{\pi}}{\sin\operatorname{coam}\frac{2Kx}{\pi}} = \frac{2K}{\pi}\cdot\tan\frac{1}{2}\operatorname{am}\frac{4Kx}{\pi}\\
-\frac{d\ln\Delta\operatorname{am}\frac{2Kx}{\pi}}{dx} &= \frac{2k^2K}{\pi}\cdot\sin\operatorname{am}\frac{2Kx}{\pi}\sin\operatorname{coam}\frac{2Kx}{\pi}
\end{aligned}$$

and

$$\frac{d\ln\sqrt{\frac{1+\sin\operatorname{am}\frac{2Kx}{\pi}}{1-\sin\operatorname{am}\frac{2Kx}{\pi}}}}{dx} = \frac{2K}{\pi}\cdot\frac{1}{\sin\operatorname{coam}\frac{2Kx}{\pi}}$$

$$\frac{d\ln\sqrt{\frac{1+k\sin\operatorname{am}\frac{2Kx}{\pi}}{1-k\sin\operatorname{am}\frac{2Kx}{\pi}}}}{dx} = \frac{2kK}{\pi}\cdot\sin\operatorname{coam}\frac{2Kx}{\pi}$$

we find the following:

$$\text{(11.)}\quad \frac{2k'K}{\pi}\cdot\frac{\cos\operatorname{am}\frac{2Kx}{\pi}}{\cos\operatorname{coam}\frac{2Kx}{\pi}} = \cot x - \frac{4q\sin 2x}{1+q} - \frac{4q^2\sin 4x}{1+q^2} - \frac{4q^3\sin 6x}{1+q^3} - \cdots$$

$$\text{(12.)}\quad \frac{2K}{\pi}\cdot\frac{\sin\operatorname{am}\frac{2Kx}{\pi}}{\sin\operatorname{coam}\frac{2Kx}{\pi}} = \tan x + \frac{4q\sin 2x}{1-q} + \frac{4q^2\sin 4x}{1+q^2} + \frac{4q^3\sin 6x}{1-q^3} + \cdots$$

$$\text{(13.)}\quad \frac{2k^2K}{\pi}\cdot\sin\operatorname{am}\frac{2Kx}{\pi}\sin\operatorname{coam}\frac{2Kx}{\pi} = \frac{8q\sin 2x}{1-q^2} + \frac{8q^3\sin 6x}{1-q^6} - \frac{8q^5\sin 10x}{1-q^{10}} - \cdots$$

$$\text{(14.)}\quad \frac{2K}{\pi\sin\operatorname{coam}\frac{2Kx}{\pi}} = \frac{1}{\cos x} + \frac{4q\cos x}{1-q} - \frac{4q^3\cos 3x}{1-q^3} - \frac{4q^5\cos 5x}{1-q^5} - \cdots$$

$$\text{(15.)}\quad \frac{2kK}{\pi}\cdot\sin\operatorname{coam}\frac{2Kx}{\pi} = \frac{4\sqrt{q}\cos x}{1-q} - \frac{4\sqrt{q^3}\cos 3x}{1-q^3} + \frac{4\sqrt{q^5}\cos 5x}{1-q^5} - \cdots$$

If in these formulas one puts $\frac{\pi}{2}-x$ instead of x, one finds:

$$\text{(16.)}\quad \frac{2k'K}{\pi}\cdot\frac{\cos\operatorname{coam}\frac{2Kx}{\pi}}{\cos\operatorname{am}\frac{2Kx}{\pi}} = \tan x - \frac{4q^2\sin 2x}{1+q} + \frac{4q^2\sin 4x}{1+q^2} - \frac{4q^3\sin 6x}{1+q^3} + \cdots$$

$$\text{(17.)}\quad \frac{2K}{\pi}\cdot\frac{\sin\operatorname{coam}\frac{2Kx}{\pi}}{\sin\operatorname{am}\frac{2Kx}{\pi}} = \cot x + \frac{4q^2\sin 2x}{1-q} - \frac{4q^2\sin 4x}{1+q^2} + \frac{4q^3\sin 6x}{1-q^3} - \cdots$$

$$\text{(18.)}\quad \frac{2K}{\pi\sin\operatorname{am}\frac{2Kx}{\pi}} = \frac{1}{\sin x} - \frac{4q\sin x}{1-q} + \frac{4q^3\sin 3x}{1-q^3} + \frac{4q^5\sin 5x}{1-q^5} + \cdots$$

$$\text{(19.)}\quad \frac{2kK}{\pi}\cdot\sin\operatorname{am}\frac{2Kx}{\pi} = \frac{4\sqrt{q}\sin x}{1-q} + \frac{4\sqrt{q^3}\sin 3x}{1-q^3} + \frac{4\sqrt{q^5}\sin 5x}{1-q^5} + \cdots$$

Formula (13.), by putting $\frac{\pi}{2} - x$ instead of x, remains unchanged.

By changing q into $-q$, from theorem I. § 37 the formulas (11.), (12.) go over into (17.), (16.); (13.) remains unchanged; from the formulas (14.), (15.), (18.), (19.) we obtain:

$$\begin{aligned}
&(20.) \quad \frac{2k'K}{\pi \cos \operatorname{am} \frac{2Kx}{\pi}} &&= \frac{1}{\cos x} - \frac{4q\cos x}{1+q} + \frac{4q^3 \cos 3x}{1+q^3} - \frac{4q^5 \cos 5x}{1+q^5} + \cdots \\
&(21.) \quad \frac{2kK}{\pi} \cdot \cos \operatorname{am} \frac{2Kx}{\pi} &&= \frac{4\sqrt{q}\cos x}{1+q} + \frac{4\sqrt{q^3}\cos 3x}{1+q^3} + \frac{4\sqrt{q^5}\cos 5x}{1+q^5} + \cdots \\
&(22.) \quad \frac{2k'K}{\pi \cos \operatorname{coam} \frac{2Kx}{\pi}} &&= \frac{1}{\sin x} - \frac{4q\sin x}{1+q} - \frac{4q^3 \sin 3x}{1+q^3} - \frac{4q^5 \sin 5x}{1+q^5} + \cdots \\
&(23.) \quad \frac{2kK}{\pi} \cdot \cos \operatorname{coam} \frac{2Kx}{\pi} &&= \frac{4\sqrt{q}\sin x}{1+q} - \frac{4\sqrt{q^3}\sin 3x}{1+q^3} + \frac{4\sqrt{q^5}\sin 5x}{1+q^5} - \cdots
\end{aligned}$$

The formulas (19.), (21.), using known expansions, can also easily be derived from those we gave in § 35 (6.), (7.):

$$\sin \operatorname{am} \frac{2Kx}{\pi} = \frac{2\pi}{kK} \sin x \left(\frac{\sqrt{q}(1+q)}{1-2q\cos x+q^2} + \frac{\sqrt{q^3}(1+q^3)}{1-2q^3\cos 2x+q^6} + \frac{\sqrt{q^5}(1+q^5)}{1-2q^5\cos 2x+q^{10}} + \cdots \right)$$

$$\cos \operatorname{am} \frac{2Kx}{\pi} = \frac{2\pi}{kK} \cos x \left(\frac{\sqrt{q}(1-q)}{1-2q\cos x+q^2} - \frac{\sqrt{q^3}(1-q^3)}{1-2q^3\cos 2x+q^6} + \frac{\sqrt{q^5}(1-q^5)}{1-2q^5\cos 2x+q^{10}} - \cdots \right)$$

From formula (9.) § 35:

$$\operatorname{am} \frac{2Kx}{\pi} = \pm x + 2\arctan \frac{(1+q)\tan x}{1-q} - 2\arctan \frac{(1+q^3)\tan x}{1-q^3} + 2\arctan \frac{(1+q^5)\tan x}{1-q^5} - \cdots$$

it also follows:

$$(24.) \qquad \operatorname{am} \frac{2Kx}{\pi} = x + \frac{2q\sin 2x}{1+q^2} + \frac{2q^2 \sin 4x}{2(1+q^4)} + \frac{2q^3 \sin 6x}{3(1+q^6)} + \cdots$$

For, the same, taking into account the ambiguous sign, can be represented this way:

$$\begin{aligned}&+x+2\arctan\frac{(1+q)t}{1-q}-2\arctan\frac{(1+q^3)t}{1-q^3}+2\arctan\frac{(1+q^5)t}{1-q^5}-\cdots\\&\quad-2x\qquad\qquad\qquad+2x\qquad\qquad\qquad-2x\qquad\qquad\qquad+\cdots\end{aligned}$$

if for the sake of brevity one puts $t=\tan x$. But it is:

$$\arctan\frac{(1+q)t}{1-q}-x=\arctan\frac{(1+q)t-(1-q)t}{1-q+(1+q)tt}=\arctan\frac{2qt}{1+tt-q(1-tt)}=\arctan\frac{q\sin 2x}{1-q\cos 2x},$$

whence it follows:

$$\operatorname{am}\frac{2Kx}{\pi}=x+2\arctan\frac{q\sin 2x}{1-q\cos 2x}-2\arctan\frac{q^3\sin 2x}{1-q^3\cos 2x}+2\arctan\frac{q^5\sin 2x}{1-q^5\cos 2x}-\cdots,$$

or because it is:

$$\arctan\frac{q\sin 2x}{1-q\cos 2x}=q\sin 2x+\frac{q^2\sin 4x}{2}+\frac{q^3\sin 6x}{3}+\cdots,$$

it is:

$$\operatorname{am}\frac{2Kx}{\pi}=x+\frac{2q\sin 2x}{1+q^2}+\frac{2q^2\sin 4x}{2(1+q^4)}+\frac{2q^3\sin 6x}{3(1+q^6)}+\cdots,$$

which is formula (24.). From its differentiation this equation results:

$$(25.)\qquad \frac{2K}{\pi}\cdot\Delta\operatorname{am}\frac{2Kx}{\pi}=1+\frac{4q\cos 2x}{1+q^2}+\frac{4q^2\cos 4x}{1+q^4}+\frac{4q^3\cos 6x}{1+q^6}+\cdots,$$

whence having put $-q$ instead of q or $\frac{\pi}{2}-x$ instead of x it also is:

$$(26.)\qquad \frac{2k'K}{\pi\Delta\operatorname{am}\frac{2Kx}{\pi}}=1-\frac{4q\cos 2x}{1+q^2}+\frac{4q^2\cos 4x}{1+q^4}-\frac{4q^3\cos 6x}{1+q^6}+\cdots$$

40.

From the propounded formulas, by putting $x = 0$ or substituting other values, the following are easily found:

$$\text{(1.)} \qquad \ln k = \ln 4\sqrt{q} - \frac{4q}{1+q} + \frac{4q^2}{2(1+q^2)} - \frac{4q^3}{3(1+q^3)} + \frac{4q^4}{4(1+q^4)} - \cdots$$

$$\text{(2.)} \qquad -\ln k' = \frac{8q}{1-q^2} + \frac{8q^3}{3(1-q^6)} + \frac{8q^5}{5(1-q^{10})} + \frac{8q^7}{7(1-q^{14})} + \cdots$$

$$\text{(3.)} \qquad \ln\frac{2K}{\pi} = \frac{4q}{1+q} + \frac{4q^3}{3(1+q^3)} + \frac{4q^5}{5(1+q^5)} + \frac{4q^7}{7(1-q^7)} + \cdots$$

And

$$\begin{aligned}
\text{(4.)} \qquad \frac{2K}{\pi} &= 1 + \frac{4q}{1-q} - \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} - \cdots \\
&= 1 + \frac{4q}{1+q^2} + \frac{4q^2}{1+q^4} + \frac{4q^3}{1+q^6} + \cdots \\
\text{(5.)} \qquad \frac{2kK}{\pi} &= \frac{4\sqrt{q}}{1-q} - \frac{4\sqrt{q^3}}{1-q^3} + \frac{4\sqrt{q^5}}{1-q^5} - \cdots \\
&= \frac{4\sqrt{q}}{1+q} + \frac{4\sqrt{q^3}}{1+q^3} + \frac{4\sqrt{q^5}}{1+q^5} + \cdots \\
\text{(6.)} \qquad \frac{2k'K}{\pi} &= 1 - \frac{4q}{1+q} - \frac{4q^3}{1+q^3} - \frac{4q^5}{1+q^5} + \cdots \\
&= 1 - \frac{4q}{1+q^2} + \frac{4q^2}{1+q^4} + \frac{4q^3}{1+q^6} + \cdots \\
\text{(7.)} \qquad \frac{2\sqrt{k'}K}{\pi} &= 1 - \frac{4q^2}{1+q^2} + \frac{4q^6}{1+q^6} - \frac{4q^{10}}{1+q^{10}} + \cdots \\
&= 1 - \frac{4q^2}{1+q^4} + \frac{4q^4}{1+q^8} - \frac{4q^6}{1+q^{12}} + \cdots \\
\text{(8.)} \qquad \frac{4KK}{\pi\pi} &= 1 + \frac{8q}{1-q} + \frac{16q^2}{1+q^2} + \frac{24q^3}{1-q^3} + \cdots \\
&= 1 + \frac{8q}{(1-q)^2} + \frac{8q^2}{(1+q^2)^2} + \frac{8q^3}{(1-q^3)^2} + \cdots
\end{aligned}$$

$$
\begin{aligned}
(9.)\qquad \frac{4kkKK}{\pi\pi} &= \frac{16q}{1-q^2} + \frac{48q^3}{1-q^6} + \frac{80q^5}{1-q^{10}} + \cdots \\
&= \frac{16q(1+q^2)}{(1-q^2)^2} + \frac{16q^3(1+q^6)}{(1-q^6)^2} + \frac{16q^5(1+q^{10})}{(1-q^{10})^2} + \cdots \\
(10.)\qquad \frac{4k'k'KK}{\pi\pi} &= 1 - \frac{8q}{1+q} + \frac{16q^2}{1+q^2} - \frac{24q^3}{1+q^3} + \cdots \\
&= 1 - \frac{8q}{(1+q)^2} + \frac{8q^2}{(1+q^2)^2} + \frac{8q^3}{(1+q^3)^2} + \cdots \\
(11.)\qquad \frac{4kk'KK}{\pi\pi} &= \frac{4\sqrt{q}}{1+q} - \frac{12\sqrt{q^3}}{1+q^3} + \frac{20\sqrt{q^5}}{1+q^5} - \cdots \\
&= \frac{4\sqrt{q}(1-q)}{(1+q)^2} - \frac{4\sqrt{q^3}(1-q^3)}{(1+q^3)^2} + \frac{4\sqrt{q^5}(1-q^5)}{(1+q^5)^2} - \cdots \\
(12.)\qquad \frac{4k'KK}{\pi\pi} &= 1 - \frac{8q^2}{1+q^2} + \frac{16q^4}{1+q^4} - \frac{24q^6}{1+q^6} + \cdots \\
&= 1 - \frac{8q^2}{(1+q^2)^2} + \frac{8q^4}{(1+q^4)^2} + \frac{8q^6}{(1+q^6)^2} + \cdots \\
(13.)\qquad \frac{4kKK}{\pi\pi} &= \frac{4\sqrt{q}}{1-q} + \frac{12\sqrt{q^3}}{1-q^3} + \frac{20\sqrt{q^5}}{1-q^5} + \cdots \\
&= \frac{4\sqrt{q}(1+q)}{(1-q)^2} + \frac{4\sqrt{q^3}(1+q^3)}{(1-q^3)^2} + \frac{4\sqrt{q^5}(1+q^5)}{(1-q^5)^2} + \cdots
\end{aligned}
$$

We represented formulas (4.)- (13.) in two ways; but the one representation easily follows from the other, if the single denominators are expanded into series. Further, we add, according to the theorems propounded in § 37, that from two of their total number, namely (4.) and (8.), one can derive them all. For, by putting $\sqrt{q}$ instead of q, because K goes over into $(1+k)K$, subtracting the result from formula (4.) (5.) results; secondly, by putting $-q$ instead of q K goes over into $k'K$, whence from the formulas (4.), (8.) formulas (6.), (10.) result; (5.) remains unchanged. By putting q^2 instead q, $k'K$ goes over into $\sqrt{k'}K$, whence from (6.), (10.) then (7.), (12.) follow. From (8.), (10.), because $kk + k'k' = 1$, (9.) results. By putting $\sqrt{q}$ instead of q, kK goes over into $2\sqrt{k}K$, whence (13.) results from (9.). By putting $-q$ instead of q, kKK goes over into $ikk'KK$, whence (11.) follows from (13.). However, series of such a kind do not seem to exist for the modulus or the complement. Having expanded the propounded formulas into a power series in q we obtain:

$$\text{(14.)}\qquad \ln k = \ln 4\sqrt{q} - 4q + 6q^2 - \frac{16}{3}q^3 + 3q^4 - \frac{24}{5}q^5 + 8q^6 - \frac{32}{7}q^7 + \frac{3}{2}q^8 - \frac{52}{9}q^9 + \frac{36}{5}q^{10} - \cdots$$

$$\text{(15.)}\qquad -\ln k' = 8q + \frac{32}{3}q^3 + \frac{48}{5}q^5 + \frac{64}{7}q^7 + \frac{104}{9}q^9 + \frac{96}{11}q^{11} + \frac{112}{13}q^{13} + \frac{192}{15}q^{15} + \cdots$$

$$\text{(16.)}\qquad \ln\frac{2K}{\pi} = 4q - 4q^2 + \frac{16}{3}q^3 - 4q^4 + \frac{24}{5}q^5 - \frac{16}{3}q^6 + \frac{32}{7}q^7 - 4q^8 + \frac{52}{9}q^9 - \frac{24}{5}q^{10} + \cdots$$

$$\text{(17.)}\qquad \frac{2K}{\pi} = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8^{10} + 8q^{13} + 4q^{16} + 8q^{17} + 4q^{18} + \cdots$$

$$\text{(18.)}\qquad \frac{2kK}{\pi} = 4\sqrt{q} + 8\sqrt{q^5} + 4\sqrt{q^9} + 8\sqrt{q^{13}} + 8\sqrt{q^{17}} + 12\sqrt{q^{25}} + 8\sqrt{q^{29}} + 8\sqrt{q^{37}} + \cdots$$

$$\text{(19.)}\qquad \frac{2k'K}{\pi} = 1 - 4q + 4q^2 + 4q^4 - 8q^5 + 4q^8 - 4q^9 + 8q^{10} - 8q^{13} + 4q^{16} - 8q^{17} + 4q^{18} + \cdots$$

$$\text{(20.)}\qquad \frac{2\sqrt{k'}K}{\pi} = 1 - 4qq2 + 4q^4 + 4q^8 - 8q^{10} + 4q^{16} - 4q^{18} + 8q^{20} - 8q^{26} + 4q^{32} - \cdots$$

$$\text{(21.)}\qquad \frac{4KK}{\pi\pi} = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + 24q^8 + \cdots$$

$$\text{(22.)}\qquad \frac{4kkKK}{\pi\pi} = 16q + 64q^3 + 96q^5 + 128q^7 + 208q^9 + 192q^{11} + 224q^{13} + 384q^{15} + \cdots$$

$$\text{(23.)}\qquad \frac{4k'k'}{\pi\pi} = 1 - 8q + 24q^2 - 32q^3 + 24q^4 - 48q^5 + 96q^6 - 64q^7 + 24q^7 - \cdots$$

$$\text{(24.)}\qquad \frac{4kk'KK}{\pi\pi} = 4\sqrt{q} - 16\sqrt{q^3} + 24\sqrt{q^5} - 32\sqrt{q^7} + 52\sqrt{q^9} - 48\sqrt{q^{11}} + 56\sqrt{q^{13}} - \cdots$$

$$\text{(25.)}\qquad \frac{4k'KK}{\pi\pi} = 1 - 8q^2 + 24q^4 - 32q^6 + 24q^8 - 48q^{10} + 96q^{12} - 64q^{14} + 24q^{16} - 104q^{18} + \cdots$$

$$\text{(26.)}\qquad \frac{4kKK}{\pi\pi} = 4\sqrt{q} + 16\sqrt{q^3} + 24\sqrt{q^5} + 32\sqrt{q^7} + 52\sqrt{q^9} + 48\sqrt{q^{11}} + 56\sqrt{q^{13}} + \cdots$$

To understand the law and the nature of these series better, we will denote them by a summation sign $\sum$ prefixed to its general term. Let us put that p is *an odd number and $\varphi(p)$ the sum of the factors of p.* Then it is:

$$\text{(27.)}\qquad \ln k = \ln 4\sqrt{q} - 4\sum \frac{\varphi(p)}{p}\left\{q^p - \frac{3q^{2p}}{p} - \frac{3}{4}q^{4p} - \frac{3}{8}q^{8p} - \frac{3}{16}q^{16p} - \cdots\right\}$$

$$\text{(28.)}\qquad -\ln k' = 8\sum \frac{\varphi(p)}{p}q^p$$

$$\text{(29.)}\qquad \ln\frac{2K}{\pi} = 4\sum \frac{\varphi(p)}{p}\left\{q^p - q^{2p} - q^{4p} - q^{8p} - q^{16p} - \cdots\right\}.$$

Further, let m be *an odd number whose prime factors are all of the form* $4a-1$, *let* n *be an odd number whose prime factors all have the form* $4a+1$, *let* $\psi(n)$ *be the number of factors of* n and finally, let l be any arbitrary number from 0 to ∞: We obtain:

$$\begin{aligned}
&(30.) \quad \frac{2K}{\pi} &&= 1 + 4\sum \psi(n) q^{2^l m^2 n} \\
&(31.) \quad \frac{2kK}{\pi} &&= \quad 4\sum \psi(n) q^{\frac{m^2 n}{2}} \\
&(32.) \quad \frac{2k'K}{\pi} &&= 1 - 4\sum \psi(n) q^{m^2 n} + 4\sum \psi(n) q^{2^{l+1} m^2 n} \\
&(33.) \quad \frac{2\sqrt{k'}K}{\pi} &&= 1 - 4\sum \psi(n) q^{2m^2 n} + 4\sum \psi(n) q^{2^{l+2} m^2 n}
\end{aligned}$$

while p again denotes an odd number, $\varphi(p)$ the sum of factors of p, it is:

$$\begin{aligned}
&(34.) \quad \frac{4KK}{\pi\pi} &&= 1 + 8\sum \varphi(p)[q^p + 3q^{2p} + 3q^{4p} + 3q^{8p} + 3q^{16p} + \cdots] \\
&(35.) \quad \frac{4kkKK}{\pi\pi} &&= \quad 16\sum \varphi(p) q^p \\
&(36.) \quad \frac{4k'k'KK}{\pi\pi} &&= 1 + 8\sum \varphi(p)[-q^p + 3q^{2p} + 3q^{4p} + 3q^{8p} + 3q^{16p} + \cdots] \\
&(37.) \quad \frac{4kk'KK}{\pi\pi} &&= \quad 4\sum (-1)^{\frac{p-1}{2}} \varphi(p) \sqrt{q^p} \\
&(38.) \quad \frac{4k'KK}{\pi\pi} &&= 1 + 8\sum \varphi(p)[-q^{2p} + 3q^{4p} + 3q^{8p} + 3q^{16p} + 3q^{32p} + \cdots] \\
&(39.) \quad \frac{4kKK}{\pi\pi} &&= \quad 4\sum \varphi(p) \sqrt{q^p}.
\end{aligned}$$

Let us demonstrate formula (27.). We found (1.):

$$\ln k = \ln 4\sqrt{q} - \frac{4q}{1+q} + \frac{4q^2}{2(1+q^2)} - \frac{4q^3}{3(1+q^4)} + \cdots,$$

which we want to put $= \ln 4\sqrt{q} + 4\sum A^{(x)} q^x$. Let x be an odd number $p = mm'$, from the general term $-\frac{q^m}{m(1+q^m)}$ $\frac{-q^p}{m}$ results, whence it is clear that it will be $A^{(p)} = -\frac{\varphi(p)}{p}$. Now, let x be an even number $= 2^l p = 2^l mm'$: from the terms

$$\frac{-q^m}{m(1+q^m)}+\frac{q^{2m}}{2m(1+q^{2m})}+\frac{q^{4m}}{4m(1+q^{4m})}++\frac{q^{8m}}{8m(1+q^{8m})}+\cdots+\frac{q^{2^l}}{2^l m(1+q^{2^l m})}$$

this equation results:

$$\frac{q^x}{m}\left\{1-\frac{1}{2}-\frac{1}{4}-\frac{1}{8}\cdots-\frac{1}{2^{l-1}}+\frac{1}{2^l}\right\}=\frac{3q^x}{2^l m},$$

whence $A^{(x)}=\frac{3\varphi(p)}{2^l p}$ what yields the propounded formula.

Let us demonstrate formula (30.). We found (4.):

$$\frac{2K}{\pi}=1+\frac{4q}{1-q}-\frac{4q^3}{1-q^3}+\frac{4q^5}{1-q^5}-\cdots=1+4\sum A^{(x)}q^x.$$

Let $B^{(x)}$ be the number of factors of x which have the form $4m+1$, $C^{(x)}$ the number of factors which have the form $4m+3$, it easily seen that $A^{(x)}=B^{(x)}-C^{(x)}$. Let $x=2^l nn'$ such that n is an odd number whose prime factors all have the form $4m+1$, n' an odd number whose prime factors all have the form $4m-1$, it is easily proved, if n' is not a square number, that it will always be $B^{(x)}-C^{(x)}=0$, and if n' is square number, that it will be $B^{(x)}-C^{(x)}=\varphi(n)$, whence formula (30.) follows:

Finally, let us prove (34.). We found (8.):

$$\frac{4KK}{\pi\pi}=1+\frac{8q}{1-q}+\frac{16q^2}{1+q^2}+\frac{24q^3}{1-q^3}+\frac{32q^4}{1+q^4}+\cdots=1+8\sum A^{(x)}q^x.$$

While x denotes an odd number it easily understood that it will be $A^{(x)}=\varphi(x)$; but if x is an even number $=2^l p$, p denoting an odd number, if m is a factor of p, from the terms

$$8\left\{\frac{mq^m}{1-q^m}+\frac{2mq^{2m}}{1+q^{2m}}+\frac{4mq^{4m}}{1+q^{4m}}+\frac{8mq^{8m}}{1+q^{8m}}+\cdots+\frac{2^l mq^{2^l m}}{1+q^{2^l m}}\right\}$$

the equation $8mq^x\left\{1-2-4-8-\cdots-2^{l-1}+2^l\right\}=24mq^x$ results, whence in this case $A^{(x)}=3\varphi(p)$ what yields the propounded formula. The remaining formulas are proven in a similar way or are deduced from these.

The expressions $\cos\operatorname{am}\frac{2Kx}{\pi}$, $\Delta\operatorname{am}\frac{2Kx}{\pi}$, $\frac{1}{\cos\operatorname{am}\frac{2Kx}{\pi}}$, having expanded them into a power series in x, have the expressions $-\frac{1}{2}\left(\frac{2K}{\pi}\right)^2$, $-\frac{1}{2}\left(\frac{2kK}{\pi}\right)^2$, $+\frac{1}{2}\left(\frac{2K}{\pi}\right)^2$ as a coefficient of x^2, respectively, whence from the formulas (21.), (25.), (20.) of the preceding paragraph we see the following equations to result:

$$\begin{aligned}
(40.)\quad k\left(\frac{2K}{\pi}\right)^3 &= \quad 4\left\{\frac{\sqrt{q}}{1+q}+\frac{9\sqrt{q^3}}{1+q^3}+\frac{25\sqrt{q^5}}{1+q^5}+\frac{49\sqrt{q^7}}{1+q^7}+\cdots\right\} \\
&= \quad 4\left\{\frac{\sqrt{q}(1+6q+q^2)}{(1-q^3)}-\frac{\sqrt{q^3}(1+6q^3+q^6)}{(1-q^3)^3}+\frac{\sqrt{q^5}(1+6q^5+q^{10})}{(1-q^5)^3}-\cdots\right\} \\
(41.)\quad k'\left(\frac{2K}{\pi}\right)^3 &= 1\ +\ 4\left\{\frac{q}{1+q}-\frac{9q^3}{1+q^3}+\frac{25q^5}{1+q^5}-\frac{49q^7}{1+q^7}+\cdots\right\} \\
&= 1\ +\ 4\left\{\frac{q(1-6q^2+q^4)}{(1+q^2)^3}-\frac{q^2(1-6q^4+q^8)}{(1+q^4)^3}+\frac{q^3(1-6q^6+q^{12})}{(1+q^3)^3}-\cdots\right\} \\
(42.)\quad kk\left(\frac{2K}{\pi}\right)^3 &= \quad 16\left\{\frac{q}{1+q^2}+\frac{4q^2}{1+q^4}+\frac{9q^3}{1+q^6}+\frac{16q^4}{1+q^8}\cdots\right\} \\
&= \quad 16\left\{\frac{q(1+q)}{(1-q)^3}-\frac{q^3(1+q^3)}{(1-q^3)^3}+\frac{q^5(1+q^5)}{(1-q^5)^3}+\cdots\right\}
\end{aligned}$$

From these, having put $-q$ instead of q, we obtain:

$$\begin{aligned}
(43.)\quad kk'k'\left(\frac{2K}{\pi}\right)^3 &= \quad 4\left\{\frac{\sqrt{q}}{1-q}-\frac{9\sqrt{q^3}}{1-q^3}+\frac{25\sqrt{q^5}}{1-q^5}-\frac{49\sqrt{q^7}}{1-q^7}+\cdots\right\} \\
(44.)\quad k'k'\left(\frac{2K}{\pi}\right)^3 &= 1-\ 4\left\{\frac{q}{1-q}-\frac{9q^3}{1-q^3}+\frac{25q^5}{1-q^5}-\frac{49q^7}{1-q^7}+\cdots\right\} \\
(45.)\quad k'kk\left(\frac{2K}{\pi}\right)^3 &= \quad 16\left\{\frac{q}{1+q^2}-\frac{4q^2}{1+q^4}+\frac{9q^3}{1-+q^6}-\frac{49q^4}{1+q^8}+\cdots\right\}.
\end{aligned}$$

Having added formulas (40.), (42.), we obtain $\left(\frac{2K}{\pi}\right)^3$; having subtracted (40.) from (43.), (41.) and (45.), we obtain $\left(\frac{2kK}{\pi}\right)^3$, $\left(\frac{2k'K}{\pi}\right)^3$, from which, having written $\sqrt{q}$, q^2 instead of q, $\left(\frac{4\sqrt{k}K}{\pi}\right)^3$, $\left(\frac{4\sqrt{k'}K}{\pi}\right)^3$ results, respectively; from $\left(\frac{4\sqrt{k}K}{\pi}\right)^3$ having put $-q$ instead of q one obtains $\left(\frac{4\sqrt{kk'}K}{\pi}\right)^3$.

At last, having put $k=\sin\vartheta$, let us expand $\vartheta=\arcsin k$. We saw, having put

$\sqrt{q}$ instead of q, that k' goes over into $\frac{1-k}{1+k}$; let us again put $-q$ instead of q, k then goes over into $\frac{ik}{k'}$ or into $i\tan\vartheta$ such that having put $i\sqrt{q}$ instead of q the expression $-\frac{\ln k'}{2i}$ is changed to:

$$-\frac{1}{2i}\ln\left(\frac{1-i\tan\vartheta}{1+i\tan\vartheta}\right)=\vartheta.$$

Hence from formula (2.):

$$-\ln k'=\frac{8q}{1-q^2}+\frac{8q^3}{3(1-q^6)}+\frac{8q^5}{5(1-q^{10})}+\frac{8q^7}{7(1-q^{14})}+\cdots$$

we find:

$$(46.)\qquad \vartheta=\arcsin k=\frac{4\sqrt{q}}{1+q}-\frac{4\sqrt{q^3}}{3(1+q^3)}+\frac{4\sqrt{q^5}}{5(1+q^5)}-\frac{4\sqrt{q^7}}{7(1+q^7)}+\cdots,$$

which is easily transformed into this one:

$$(47.)\qquad \frac{\vartheta}{4}=\arctan\sqrt{q}-\arctan\sqrt{q^3}+\arctan\sqrt{q^5}-\arctan\sqrt{q^7}+\cdots,$$

which is to be counted among the most elegant formulas.

41.

Let us multiply the equation exhibited above:

$$\frac{2kK}{\pi}\sin\operatorname{am}\frac{2Kx}{\pi}=\frac{4\sqrt{q}\sin x}{1-q}+\frac{4\sqrt{q^3}\sin 3x}{1-q^3}+\frac{4\sqrt{q^5}\sin 5x}{1-q^5}+\cdots$$

by itself. Having substituted the expression

$$\cos(m-n)x-\cos(m+n)x$$

for $2\sin mx\sin nx$ everywhere the square takes on the following form:

$$\left(\frac{2kK}{\pi}\right)^2\sin^2\operatorname{am}\frac{2Kx}{\pi}=A+A'\cos 2x+A''\cos 4x+A'''\cos 6x+\cdots$$

It is found:

$$A = \frac{8q}{(1-q)^2} + \frac{8q^3}{(1-q^3)^2} + \frac{8q^5}{(1-q^5)^2} + \cdots$$

Further, it is:

$$A^{(n)} = 16B^{(n)} - 8C^{(n)} = 8[2B^{(n)} - C^{(n)}],$$

if it is put:

$$B^{(n)} = \frac{q^{n+1}}{(1-q)(1-q^{2n+1})} + \frac{q^{n+3}}{(1-q^3)(1-q^{2n+3})} + \frac{q^{n+5}}{(1-q^5)(1-q^{2n+5})} + \text{etc. to infinity}$$

$$C^{(n)} = \frac{q^n}{(1-q)(1-q^{2n-1})} + \frac{q^n}{(1-q^3)(1-q^{2n-3})} + \frac{q^n}{(1-q^5)(1-q^{2n-5})} + \cdots + \frac{q^n}{(1-q^{2n-1})(1-q)}.$$

Now, because it is:

$$\frac{q^{m+n}}{(1-q^m)(1-q^{2m+n})} = \frac{q^n}{1-q^{2n}}\left\{\frac{q^m}{1-q^m} - \frac{q^{2n+m}}{1-q^{2n+m}}\right\}$$

it is:

$$B^{(n)} = \left\{\begin{array}{l} \dfrac{q^n}{1-q^{2n}}\left\{\dfrac{q}{1-q} + \dfrac{q^3}{1-q^3} + \dfrac{q^5}{1-q^5} + \cdots\right\} \\ -\dfrac{q^n}{1-q^{2n}}\left\{\dfrac{q^{2n+1}}{1-q^{2n+1}} + \dfrac{q^{2n+3}}{1-q^{2n+3}} + \dfrac{q^{2n+5}}{1-q^{2n+5}} + \cdots\right\} \end{array}\right\}$$

or having cleared the terms which cancel each other:

$$B^{(n)} = \frac{q^n}{1-q^{2n}}\left\{\frac{q}{1-q} + \frac{q^3}{1-q^3} + \cdots + \frac{q^{2n-1}}{1-q^{2n-1}}\right\}.$$

Further, it is:

$$\frac{q^n}{(1-q^m)(1-q^{2n-m})} = \frac{q^n}{1-q^{2n}}\left\{\frac{q^m}{1-q^m} + \frac{q^{2n-m}}{1-q^{2n-m}} + 1\right\},$$

whence:

$$C^{(n)} = \frac{nq^n}{1-q^{2n}} + \frac{2q^n}{1-q^{2n}}\left\{\frac{q}{1-q} + \frac{q^3}{1-q^3} + \cdots + \frac{q^{2n-1}}{1-q^{2n-1}}\right\}.$$

Hence finally this equation results:

$$A^{(n)} = 8[2B^{(n)} - C^{(n)}] = \frac{-8nq^n}{1-q^{2n}},$$

whence now:

$$(1.)\qquad \left(\frac{2kK}{\pi}\right)^2 \sin^2 \operatorname{am} \frac{2Kx}{\pi} = A - 8\left\{\frac{q\cos 2x}{1-q^2} + \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} + \cdots\right\}.$$

In like manner from (1.) it is also found:

$$(2.)\qquad \left(\frac{2kK}{\pi}\right)^2 \cos^2 \operatorname{am} \frac{2Kx}{\pi} = B + 8\left\{\frac{q\cos 2x}{1-q^2} + \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} + \cdots\right\},$$

if:

$$\begin{aligned} A &= 8\left\{\frac{q}{(1-q)^2} + \frac{q^3}{(1-q^3)} + \frac{q^5}{(1-q^5)} + \cdots\right\} \\ B &= 8\left\{\frac{q}{(1+q)^2} + \frac{q^3}{(1+q^3)} + \frac{q^5}{(1+q^5)} + \cdots\right\}. \end{aligned}$$

Applying a known theorem of integral calculus, if

$$\varphi(x) = A + A'\cos 2x + A''\cos 4x + A'''\cos 6x + \cdots,$$

the constant or first term is found to be:

$$A = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \varphi(x)dx,$$

whence in this case we obtain:

$$\begin{aligned} A &= \frac{2}{\pi}\left(\frac{2kK}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} \sin^2 \operatorname{am} \frac{2Kx}{\pi} dx \\ B &= \frac{2}{\pi}\left(\frac{2kK}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} \cos^2 \operatorname{am} \frac{2Kx}{\pi} dx. \end{aligned}$$

Following Legendre, let us put:

$$E^{\mathrm{I}} = \int_0^{\frac{\pi}{2}} d\varphi \Delta(\varphi) = \frac{2K}{\pi} \int_0^{\frac{\pi}{2}} dx \Delta^2 \operatorname{am} \frac{2Kx}{\pi},$$

it will be:

$$\begin{aligned} A &= \frac{2K}{\pi} \cdot \frac{2K}{\pi} - \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} \\ B &= \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} - \left(\frac{2k'K}{\pi}\right)^2. \end{aligned}$$

Hence, because having changed q into $-q$ A goes over into $-q$, K into $k'K$, it follows that at the same time E^{I} goes over into $\frac{E^{\mathrm{I}}}{k'}$.

Finally, let us add that from formula (1.) it follows:

$$(3.) \quad kk\left(\frac{2K}{\pi}\right)^4 = 16\left\{\frac{q}{1-q^2} + \frac{2^3q^2}{1-q^4} + \frac{3^3q^3}{1-q^6} + \frac{4^3q^4}{1-q^8} + \cdots\right\}$$
$$= 16\left\{\frac{q(1+4q+q^2)}{(1-q)^4} + \frac{q^3(1+4q^3+q^6)}{(1-q^3)^4} + \frac{q^5(1+4q^5+q^{10})}{(1-q^5)} + \cdots\right\},$$

whence having changed q to $-q$ it also is:

$$(4.) \quad k^2k\left(\frac{2K}{\pi}\right)^4 = 16\left\{\frac{q}{1-q^2} - \frac{2^3q^2}{1-q^4} + \frac{3^3q^3}{1-q^6} - \frac{4^3q^4}{1-q^8} + \cdots\right\}$$
$$= 16\left\{\frac{q(1-4q+q^2)}{(1+q)^4} + \frac{q^3(1+4q^3+q^6)}{(1+q^3)^4} + \frac{q^5(1-4q^5+q^{10})}{(1+q^5)} + \cdots\right\}.$$

Having subtracted formula (4.) from (3.) this expression results:

$$(5.) \quad \left(\frac{2kK}{\pi}\right)^4 = 256\left\{\frac{q^2}{1-q^4} + \frac{2^3q^4}{1-q^8} + \frac{3^3q^6}{1-q^{12}} + \frac{4^3q^8}{1-q^{16}} + \cdots\right\}$$
$$= 256\left\{\frac{q^2(1+4q^2+q^4)}{(1-q^2)^4} + \frac{q^6(1+4q^6+q^{12})}{(1-q^6)^4} + \frac{q^{10}(1+4q^{10}+q^{20})}{(1-q^{10})} + \cdots\right\},$$

which one obtains also from (3.) having changed q into q^2.

42.

Using a similar method as formula (1.) of the preceding § was found we could have investigated, how to expand the expression

$$\frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2 \operatorname{am} \frac{2Kx}{\pi}}$$

into a series, if formula (18.) in § 39 is squared. This is, however, achieved more easily starting from (1.) of § 41 having considered the following.

For, having differentiated the formula:

$$\frac{d \ln \sin \operatorname{am} \frac{2Kx}{\pi}}{dx} = \frac{2K}{\pi} \cdot \frac{\sqrt{1-(1+kk)\sin^2 \operatorname{am} \frac{2Kx}{\pi} + kk \sin^4 \operatorname{am} \frac{2Kx}{\pi}}}{\sin \operatorname{am} \frac{2Kx}{\pi}}$$

one more time and having done the reductions, we obtain:

$$(1.) \qquad \frac{d^2 \ln \sin \operatorname{am} \frac{2Kx}{\pi}}{dx^2} = \left(\frac{2K}{\pi}\right)^2 \left\{ kk \sin^2 \operatorname{am} \frac{2Kx}{\pi} - \frac{1}{\sin^2 \operatorname{am} \frac{2Kx}{\pi}} \right\}.$$

But in § 39 (6.) we already found:

$$\ln \sin \operatorname{am} \frac{2Kx}{\pi} = \ln\left(\frac{2\sqrt[4]{q}}{\sqrt{k}}\right) + \ln \sin x + 2\left\{ \frac{q\cos 2x}{1+q} + \frac{q^2 \cos 4x}{2(1+q^2)} + \frac{q^3 \cos 6x}{3(1+q^3)} + \cdots \right\},$$

whence it is:

$$\frac{d^2 \ln \sin \operatorname{am} \frac{2Kx}{\pi}}{dx^2} = -\frac{1}{\sin^2 x} - 8\left\{ \frac{q\cos 2x}{1+q} + \frac{2q^2 \cos 4x}{1+q^2} + \frac{3q^3 \cos 6x}{1+q^3} + \cdots \right\}.$$

Further, it is § 41 (1.):

$$\left(\frac{2kK}{\pi}\right)^2 \sin^2 \operatorname{am} \frac{2Kx}{\pi} = \frac{2K}{\pi} \cdot \frac{2K}{\pi} - \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} - 8\left\{ \frac{q\cos 2x}{1-q^2} + \frac{2q^2 \cos 4x}{1-q^4} + \frac{3q^3 \cos 6x}{1-q^6} + \cdots \right\},$$

whence, because from formula (1.) it is:

$$\frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2 \operatorname{am} \frac{2Kx}{\pi}} = \left(\frac{2kK}{\pi}\right)^2 \sin^2 \operatorname{am} \frac{2Kx}{\pi} - \frac{d^2 \ln \sin \operatorname{am} \frac{2Kx}{\pi}}{dx^2},$$

what we are looking for results, namely:

$$(2.)\quad \frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2 \operatorname{am} \frac{2Kx}{\pi}} = \frac{2K}{\pi} \cdot \frac{2K}{\pi} - \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} + \frac{1}{\sin^2 x} - 8 \left\{ \frac{q \cos 2x}{1-q^2} + \frac{2q^2 \cos 4x}{1-q^4} + \frac{3q^3 \cos 6x}{1-q^6} + \cdots \right\}$$

Having changed q to $-q$ at the same time as x was changed to $\frac{\pi}{2} - x$, whence K goes over into $k'K$, E^{I} into $\frac{E^{\mathrm{I}}}{k'}$ (§ 41), $\sin \operatorname{am} \frac{2Kx}{\pi}$ into $\cos \operatorname{am} \frac{2Kx}{\pi}$, from (2.) this expression results:

$$(3.)\quad \frac{\left(\frac{2k'K}{\pi}\right)^2}{\cos^2 \operatorname{am} \frac{2Kx}{\pi}} = \left(\frac{2k'K}{\pi}\right)^2 - \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} + \frac{1}{\cos^2 x} + 8 \left\{ \frac{q^2 \cos 2x}{1-q^2} - \frac{2q^4 \cos 4x}{1-q^4} + \frac{3q^6 \cos 6x}{1-q^6} - \cdots \right\}$$

To these I add these formulas following directly from § 41. (1.):

$$(4.)\quad \left(\frac{2K}{\pi}\right)^2 \Delta^2 \operatorname{am} \frac{2Kx}{\pi} = \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} + 8 \left\{ \frac{q \cos 2x}{1-q^2} + \frac{2q^2 \cos 4x}{1-q^4} + \frac{3q^3 \cos 6x}{1-q^6} + \cdots \right\}$$

$$(5.)\quad \left(\frac{2k'K}{\pi}\right)^2 \frac{1}{\Delta^2 \operatorname{am} \frac{2Kx}{\pi}} = \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} - 8 \left\{ \frac{q \cos 2x}{1-q^2} - \frac{2q^2 \cos 4x}{1-q^4} + \frac{3q^3 \cos 6x}{1-q^6} - \cdots \right\}$$

of which (5.) follows from (4.) having changed x into $\frac{\pi}{2} - x$ or q into $-q$.

Having put $y = \sin \operatorname{am} \frac{2Kx}{\pi}$, $\sqrt{(1-y^2)(1-k^2y^2)} = R$ it is:

$$\begin{aligned}
\frac{dy}{dx} &= \quad \left(\frac{2K}{\pi}\right)\cdot R \\
\frac{d^2y}{dx^2} &= -\left(\frac{2K}{\pi}\right)^2 y(1+k^2-2k^2y^2) \\
\frac{d^3y}{dx^3} &= -\left(\frac{2K}{\pi}\right)^3 (1+k^2-6k^2y^2)R \\
\frac{d^4y}{dx^4} &= \quad \left(\frac{2K}{\pi}\right)^4 y(1+14k^2+k^4-20k^3(1+k^2)y^2+24k^4y^4) \\
\frac{d^5y}{dx^5} &= \quad \left(\frac{2K}{\pi}\right)^5 (1+14k^2+k^4-60k^2(1+k^2)y^2+120k^4y^4)R \\
& \qquad \text{etc.} \qquad\qquad \text{etc.,}
\end{aligned}$$

whence:

$$y = \sin\mathrm{am}\,\frac{2Kx}{\pi} = \frac{2Kx}{\pi} - \frac{1+k^2}{2\cdot 3}\left(\frac{2Kx}{\pi}\right)^3 + \frac{1+14k^2+k^4}{2\cdot 3\cdot 4\cdot 5}\left(\frac{2Kx}{\pi}\right)^5 - \cdots$$

and hence:

$$\frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2\mathrm{am}\,\frac{2Kx}{\pi}} = \frac{1}{x^2} + \frac{1+k^2}{3}\left(\frac{2K}{\pi}\right)^2 + \frac{1-k^2+k^4}{15}\left(\frac{2K}{\pi}\right)^2 x^2 + \cdots,$$

after having compared which formula to (2.) it is found:

$$\frac{1+k^2}{3}\left(\frac{2K}{\pi}\right)^2 = \frac{1}{3} + \left(\frac{2K}{\pi}\right)^2 - \frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} - 8\left\{\frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \cdots\right\},$$

or

$$(6.) \qquad \frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \frac{4q^8}{1-q^8} + \cdots = \frac{1+\left(\frac{2K}{\pi}\right)^2(2-k^2) - 3\frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi}}{2\cdot 3\cdot 4}.$$

Further, it is:

$$\frac{1-k^2+k^4}{15}\left(\frac{2K}{\pi}\right)^4 = \frac{1}{15}+16\left\{\frac{q^2}{1-q^2}+\frac{2^3q^4}{1-q^4}+\frac{3^3q^6}{1-q^6}+\frac{4^3q^8}{1-q^8}+\cdots\right\}$$

or because it is $15 = 2\cdot 2^3 - 1$:

$$\begin{aligned}(1-k^2+k^4)\left(\frac{2K}{\pi}\right)^4 &= 1+2\cdot 16\left\{\frac{2^3q^2}{1-q^2}+\frac{4^3q^4}{1-q^4}+\frac{6^3q^6}{1-q^6}+\frac{8^3q^8}{1-q^8}+\cdots\right\}\\ &\quad -1\cdot 16\left\{\frac{q^2}{1-q^2}+\frac{2^3q^4}{1-q^4}+\frac{3^3q^6}{1-q^6}+\frac{4^3q^8}{1-q^8}+\cdots\right\}.\end{aligned}$$

From this formula subtract the following (3.) of § 41:

$$k^2\left(\frac{2K}{\pi}\right)^4 = 16\left\{\frac{q}{1-q^2}+\frac{2^3q^2}{1-q^4}+\frac{3^3q^3}{1-q^6}+\frac{4^3q^4}{1-q^8}+\cdots\right\}$$

the residue is:

$$(7.)\qquad \left(\frac{2k'K}{\pi}\right)^4 = 1-16\left\{\frac{q}{1-q}-\frac{2^3q^2}{1-q^2}+\frac{3^3q^3}{1-q^3}-\frac{4^3q^4}{1-q^4}+\cdots\right\},$$

whence having changed q to $-q$ it also is:

$$(8.)\qquad \left(\frac{2K}{\pi}\right)^4 = 1+16\left\{\frac{q}{1+q}+\frac{2^3q^2}{1-q^2}+\frac{3^3q^3}{1+q^3}+\frac{4^3q^4}{1-q^4}+\cdots\right\},$$

which formulas were more difficult to find. If one combines them with those we found above, one has now expanded the first four powers of $\frac{2K}{\pi}$, $\frac{2kK}{\pi}$ into a beautiful series.

2.2 General Formulas for the expansion of the Functions $\sin^n \operatorname{am} \frac{2Kx}{\pi}$, $\frac{1}{\sin^n \operatorname{am} \frac{2Kx}{\pi}}$ into a series of sines and cosines of multiples of x

43.

Having found the expansions of the functions

$$\sin\operatorname{am}\frac{2Kx}{\pi},\quad \sin^2\operatorname{am}\frac{2Kx}{\pi},\quad \frac{1}{\sin\operatorname{am}\frac{2Kx}{\pi}},\quad \frac{1}{\sin^2\operatorname{am}\frac{2Kx}{\pi}},$$

this automatically raises the question about the expansion of the powers of

$$\sin\operatorname{am}\frac{2Kx}{\pi},\quad \frac{1}{\sin\operatorname{am}\frac{2Kx}{\pi}}.$$

There is an easy way via analytic geometry following which having found the expansion of $\sin x$, $\cos x$ you can proceed to the expansion of the expressions $\cos^n x$, $\sin^n x$; this is certainly achieved by means of the known formulas expressing $\sin^n x$ and $\cos^n x$ as linear combinations of sines and cosines of multiples of x. But since in the theory of elliptic function such an auxiliary tool does not exist, one will have to use another way we will explain in the following.

Having differentiated the formula, which is obvious from the elements,:

$$\frac{d\sin^n\operatorname{am}u}{du} = n\sin^{n-1}\operatorname{am}u\sqrt{1-(1+k^2)\sin^2\operatorname{am}u+k^2\sin^4\operatorname{am}u}$$

one more time this expression results:

$$(1.)\quad \frac{d^2\sin^n\operatorname{am}u}{du^2} = n(n-1)\sin^{n-2}\operatorname{am}u - n^2(1+k^2)\sin^n\operatorname{am}u + n(n+1)k^2\sin^{n+2}\operatorname{am}u.$$

Having successively put $n = 1, 3, 5, 7\cdots$, $n = 2, 4, 6, 8, \cdots$ from this then form two series of equations:

I.

$$\begin{aligned}
\frac{d^2\sin\operatorname{am}u}{du^2} &= \phantom{6\sin\operatorname{am}u} -1(1+k^2)\sin\operatorname{am}u + 2k^2\sin^3\operatorname{am}u\\
\frac{d^2\sin^3\operatorname{am}u}{du^2} &= 6\sin\operatorname{am}u - 9(1+k^2)\sin^3\operatorname{am}u + 12k^2\sin^5\operatorname{am}u\\
\frac{d^2\sin^5\operatorname{am}u}{du^2} &= 20\sin^3\operatorname{am}u - 25(1+k^2)\sin^5\operatorname{am}u + 30k^2\sin^7\operatorname{am}u\\
\frac{d^2\sin^7\operatorname{am}u}{du^2} &= 42\sin^5\operatorname{am}u - 49(1+k^2)\sin^7\operatorname{am}u + 56k^2\sin^9\operatorname{am}u\\
&\text{etc.}\qquad\qquad \text{etc.}
\end{aligned}$$

II.

$$
\begin{aligned}
\frac{d^2 \sin^2 \operatorname{am} u}{du^2} &= 2 \qquad\qquad\quad - 4(1+k^2)\ \sin \operatorname{am} u\ + 6k^2 \sin^4 \operatorname{am} u \\
\frac{d^2 \sin^4 \operatorname{am} u}{du^2} &= 12 \sin^2 \operatorname{am} u - 16(1+k^2) \sin^4 \operatorname{am} u + 20k^2 \sin^6 \operatorname{am} u \\
\frac{d^2 \sin^6 \operatorname{am} u}{du^2} &= 30 \sin^4 \operatorname{am} u - 36(1+k^2) \sin^6 \operatorname{am} u + 42k^2 \sin^8 \operatorname{am} u \\
\frac{d^2 \sin^8 \operatorname{am} u}{du^2} &= 56 \sin^6 \operatorname{am} u - 64(1+k^2) \sin^8 \operatorname{am} u + 72k^2 \sin^{10} \operatorname{am} u \\
& \quad \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}
$$

From equations I., II. having put $\Pi n = 1 \cdot 2 \cdot 3 \cdots n$ one successively finds:

I. a.

$$
\begin{aligned}
\Pi 2 \cdot k^2 \sin^3 \operatorname{am} u &= \frac{d^2 \sin \operatorname{am} u}{du^2} + (1+k^2) \sin \operatorname{am} u \\
\Pi 4 \cdot k^4 \sin^5 \operatorname{am} u &= \frac{d^4 \sin \operatorname{am} u}{du^4} + 10(1+k^2) \frac{\sin \operatorname{am} u}{du^2} + 3(3+2k^2+3k^4) \sin \operatorname{am} u \\
\Pi 6 \cdot k^6 \sin^7 \operatorname{am} u &= \frac{d^6 \sin \operatorname{am} u}{du^6} + 35(1+k^2) \frac{d^4 \sin \operatorname{am} u}{du^4} + 7(37+38k^2+37k^4) \frac{d^2 \sin \operatorname{am} u}{du^2} \\
& \quad + 45(5+3k^2+3k^4+5k^6) \sin \operatorname{am} u \\
\Pi 8 \cdot k^8 \sin^9 \operatorname{am} u &= \frac{d^8 \sin \operatorname{am} u}{du^8} + 84(1+k^2) \frac{d^6 \sin \operatorname{am} u}{du^6} + 42(47+58k^2+47k^4) \frac{d^4 \sin \operatorname{am} u}{du^4} \\
& \quad + 4(3229+3315k^2+3315k^4+3229k^6) \frac{d^2 \sin \operatorname{am} u}{du^2} \\
& \quad + 315(35+20k^2+18k^4+20k^6+35k^8) \sin \operatorname{am} u \\
& \qquad \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}
$$

$$\Pi 3\cdot k^2\sin^4\mathrm{am}\,u=\frac{d^2\sin^2\mathrm{am}\,u}{du^2}+4(1+k^2)\sin^2\mathrm{am}\,u-2$$
$$\Pi 5\cdot k^4\sin^6\mathrm{am}\,u=\frac{d^4\sin^2\mathrm{am}\,u}{du^4}+20(1+k^2)\frac{d^2\sin^2\mathrm{am}\,u}{du^2}+8(8+7k^2+8k^4)\sin^2\mathrm{am}\,u-32(1+k^2)$$
$$\Pi 7\cdot k^6\sin^8\mathrm{am}\,u=\frac{d^6\sin^2\mathrm{am}\,u}{du^6}+56(1+k^2)\frac{d^4\sin^2\mathrm{am}\,u}{du^4}+112(7+8k^2+7k^4)\frac{d^2\sin^2\mathrm{am}\,u}{du^2}$$
$$+128(18+15k^2+15k^4+18k^6)\sin^2\mathrm{am}\,u-48(24+23k^2+24k^4)$$
$$\text{etc.}\qquad\qquad \text{etc.}$$

So we see that we can put in general:

$$(2.)\qquad \Pi(2n)\cdot k^{2n}\sin^{2n+1}\mathrm{am}\,u$$
$$=\frac{d^{2n}\sin\mathrm{am}\,u}{du^{2n}}+A_n^{(1)}\frac{d^{2n-2}\sin\mathrm{am}\,u}{du^{2n-2}}+A_n^{(2)}\frac{d^{2n-4}\sin\mathrm{am}\,u}{du^{2n-4}}+\cdots+A_n^{(n)}\sin\mathrm{am}\,u$$
$$(3.)\qquad \Pi(2n-1)\cdot k^{2n-2}\sin^{2n}\mathrm{am}\,u$$
$$=\frac{d^{2n-2}\sin^2\mathrm{am}\,u}{du^{2n-2}}+B_n^{(1)}\frac{d^{2n-4}\sin^2\mathrm{am}\,u}{du^{2n-4}}+B_n^{(2)}\frac{d^{2n-6}\sin^2\mathrm{am}\,u}{du^{2n-6}}+\cdots+B_n^{(n-1)}\sin^2\mathrm{am}\,u+B_n^{(n)},$$

where $A_n^{(m)}$, $B_n^{(m)}$ denote polynomial functions of k^2 of $m-$th order except for $B_n^{(n)}$ which is of $(n-2)-$th order. Further, from the general formula from which we started:

$$\frac{d^2\sin^n\mathrm{am}\,u}{du^2}=n(n-1)\sin^{n-2}\mathrm{am}\,u-n^2(1+k^2)\sin^n\mathrm{am}\,u+n(n+1)k^2\sin^{n+2}\mathrm{am}\,u$$

it is clear that it will be:

$$(4.)\qquad A_n^{(m)}=A_{n-1}^{(m)}+(2n-1)^2(1+k^2)A_{n-1}^{(m-1)}-(2n-2)^2(2n-1)(2n-3)k^2A_{n-2}^{(m-2)}$$
$$(5.)\qquad B_n^{(m)}=B_{n-1}^{(m)}+(2n-2)^2(1+k^2)B_{n-1}^{(m-1)}-(2n-3)^2(2n-2)(2n-4)k^2B_{n-2}^{(m-2)},$$

in which formulas, if $m>n$, one has to put $A_n^{(m)}=0$, $B_n^{(m)}=0$.

Having changed u into $u+iK'$, since $\sin\mathrm{am}\,u$ goes over into $\frac{1}{k\sin\mathrm{am}\,u}$, one will be able to put $\frac{1}{\sin\operatorname{am}u}$ instead of $\sin\operatorname{am}u$ in the propounded formulas, whence the following formulas result:

$$\begin{aligned}
\frac{\Pi 2}{\sin^3\operatorname{am}u} &= \frac{d^2}{du^2}\frac{1}{\sin\operatorname{am}u} + 1(1+k^2)\frac{1}{\sin\operatorname{am}u} \\
\frac{\Pi 3}{\sin^4\operatorname{am}u} &= \frac{d^2}{du^2}\frac{1}{\sin^2\operatorname{am}u} + 4(1+k^2)\frac{1}{\sin^2\operatorname{am}u} - 2k^2 \\
\frac{\Pi 4}{\sin^5\operatorname{am}u} &= \frac{d^4}{du^4}\frac{1}{\sin\operatorname{am}u} + 10(1+k^2)\frac{d^2}{du^2}\frac{1}{\sin\operatorname{am}u} + \frac{3(3+2k^2+3k^4)}{\sin\operatorname{am}u} \\
\frac{\Pi 5}{\sin^6\operatorname{am}u} &= \frac{d^4}{du^4}\frac{1}{\sin^2\operatorname{am}u} + 20(1+k^2)\frac{d^2}{du^2}\frac{1}{\sin^2\operatorname{am}u} + \frac{8(8+7k^2+8k^4)}{\sin^2\operatorname{am}u} - 32k^2(1+k^2) \\
&\qquad\qquad \text{etc.} \qquad\qquad\qquad \text{etc.,}
\end{aligned}$$

and in general

$$(6.)\qquad \frac{\Pi(2n)}{\sin^{2n+1}\operatorname{am}u}$$

$$= \frac{d^{2n}}{du^{2n}}\frac{1}{\sin\operatorname{am}u} + A_n^{(1)}\frac{d^{2n-2}}{du^{2n-2}}\frac{1}{\sin\operatorname{am}u} + A_n^{(2)}\frac{d^{2n-4}}{du^{2n-4}}\frac{1}{\sin\operatorname{am}u} + \cdots + A_n^{(n)}\frac{1}{\sin\operatorname{am}u}$$

$$(7.)\qquad \frac{\Pi(2n-1)}{\sin^{2n}\operatorname{am}u}$$

$$= \frac{d^{2n-2}}{du^{2n-2}}\frac{1}{\sin^2\operatorname{am}u} + B_n^{(1)}\frac{d^{2n-4}}{du^{2n-4}}\frac{1}{\sin^2\operatorname{am}u} + B_n^{(2)}\frac{d^{2n-6}}{du^{2n-6}}\frac{1}{\sin^2\operatorname{am}u} + \cdots + B_n^{(n)}\frac{1}{\sin^2\operatorname{am}u}$$

44.

Because it was found in the preceding §, if one puts $u = \frac{2Kx}{\pi}$, that the expressions

$$\sin^n\operatorname{am}\frac{2Kx}{\pi}, \quad \frac{1}{\sin^n\operatorname{am}\frac{2Kx}{\pi}}$$

can be expressed as a linear combination of these:

$$\sin\operatorname{am}\frac{2Kx}{\pi}, \quad \sin^2\operatorname{am}\frac{2Kx}{\pi}, \quad \frac{1}{\sin\operatorname{am}\frac{2Kx}{\pi}}, \quad \frac{1}{\sin^2\operatorname{am}\frac{2Kx}{\pi}}$$

and its differentials taken with respect to the argument u or x, from their expansion into a series of sines and cosines of multiples of the argument x

the corresponding expansions immediately follow.

This way we obtain:

I.

From the formula:

$$\frac{2kK}{\pi}\sin\operatorname{am}\frac{2Kx}{\pi} = 4\left\{\frac{\sqrt{q}\sin x}{1-q} + \frac{\sqrt{q^3}\sin 3x}{1-q^3} + \frac{\sqrt{q^5}\sin 5x}{1-q^5} + \cdots\right\}$$

we obtain the following:

$$2\left(\frac{2kK}{\pi}\right)^3 \sin^3\operatorname{am}\frac{2Kx}{\pi}$$

$$\begin{aligned}
&2\left(\frac{2kK}{\pi}\right)^3 \sin^3\operatorname{am}\frac{2Kx}{\pi}\\
&= 4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 1^2\right\}\frac{\sqrt{q}\sin x}{1-q}\\
&+ 4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 3^2\right\}\frac{\sqrt{q^3}\sin 3x}{1-q^3}\\
&+ 4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 5^2\right\}\frac{\sqrt{q^5}\sin 5x}{1-q^5}\\
&+ \cdots
\end{aligned}$$

$$\begin{aligned}
&2\cdot 3\cdot 4\left(\frac{2kK}{\pi}\right)^5 \sin^5\operatorname{am}\frac{2Kx}{\pi}\\
&= 4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 1^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 1^4\right\}\frac{\sqrt{q}\sin x}{1-q}\\
&+ 4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 3^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 3^4\right\}\frac{\sqrt{q^3}\sin 3x}{1-q^3}\\
&+ 4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 5^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 5^4\right\}\frac{\sqrt{q^5}\sin 5x}{1-q^5}\\
&+ \cdots
\end{aligned}$$

etc. etc.

II.

From the formula:

$$\left(\frac{2kK}{\pi}\right)^2 \sin^2 \operatorname{am} \frac{2Kx}{\pi} = \frac{2K}{\pi} \cdot \frac{2K}{\pi} - \frac{2K}{\pi} \cdot \frac{2E^{\mathrm{I}}}{\pi} - 4\left\{\frac{2q\cos 2x}{1-q^2} + \frac{4q^2\cos 4x}{1-q^4} + \frac{6q^3\cos 6x}{1-q^6} + \cdots\right\}$$

you obtain the following:

$$\begin{aligned}
&2\cdot 3\left(\frac{2kK}{\pi}\right)^4 \sin^4 \operatorname{am}\frac{2Kx}{\pi}\\
&= 4(1+k^2)\left(\frac{2K}{\pi}\right)^3\left(\frac{2K}{\pi} - \frac{2E^{\mathrm{I}}}{\pi}\right) - 2k^2\left(\frac{2K}{\pi}\right)^4\\
&- 4\left\{2\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 2^3\right\} - \frac{q\cos 2x}{1-q^2}\\
&- 4\left\{2\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 4^3\right\} - \frac{q^2\cos 4x}{1-q^4}\\
&- 4\left\{2\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 6^3\right\} - \frac{q^3\cos 6x}{1-q^6}\\
&- \cdots
\end{aligned}$$

$$\begin{aligned}
&2\cdot 3\cdot 4\cdot 5\left(\frac{2kK}{\pi}\right)^6 \sin^6 \operatorname{am}\frac{2Kx}{\pi}\\
&= 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^5\left(\frac{2K}{\pi} - \frac{2E^{\mathrm{I}}}{\pi}\right) - 32k^2(1+k^2)\left(\frac{2K}{\pi}\right)^3\\
&- 4\left\{2\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4 - 2^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 2^5\right\}\frac{q\cos 2x}{1-q^2}\\
&- 4\left\{4\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4 - 4^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 4^5\right\}\frac{q^2\cos 4x}{1-q^4}\\
&- 4\left\{6\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4 - 6^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 6^5\right\}\frac{q^3\cos 6x}{1-q^6}\\
&- \cdots
\end{aligned}$$

etc. etc.

III.

From the formula:

$$\frac{\frac{2K}{\pi}}{\sin \operatorname{am} \frac{2Kx}{\pi}} = \frac{1}{\sin x} + \frac{4q \sin x}{1-q} + \frac{4q^3 \sin 3x}{1-q^3} + \frac{4q^5 \sin 5x}{1-q^5} + \text{etc.}$$

you obtain the following:

$$\begin{aligned}
&\frac{2\left(\frac{2K}{\pi}\right)^3}{\sin^3 \operatorname{am} \frac{2Kx}{\pi}} \\
&= (1+k^2)\left(\frac{2K}{\pi}\right)^2 \frac{1}{\sin x} + \frac{d^2}{dx^2} \\
&+ 4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 1^2\right\} \frac{q \sin x}{1-q} \\
&+ 4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 3^2\right\} \frac{q^3 \sin 3x}{1-q^3} \\
&+ 4\left\{(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 5^2\right\} \frac{q^5 \sin 5x}{1-q^5} \\
&+ \cdots
\end{aligned}$$

$$\begin{aligned}
&\frac{2\cdot 3\cdot 4\left(\frac{2K}{\pi}\right)^5}{\sin^5 \operatorname{am} \frac{2Kx}{\pi}} \\
&= \frac{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4}{\sin x} + 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 \frac{d^2}{dx^2}\frac{1}{\sin x} + \frac{d^4}{dx^4}\frac{1}{\sin x} \\
&+ 4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 1^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 1^4\right\} \frac{q \sin x}{1-q} \\
&+ 4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 3^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 3^4\right\} \frac{q^3 \sin 3x}{1-q^3} \\
&+ 4\left\{3(3+2k^2+3k^4)\left(\frac{2K}{\pi}\right)^4 - 5^2\cdot 10(1+k^2)\left(\frac{2K}{\pi}\right)^2 + 5^4\right\} \frac{q^5 \sin 5x}{1-q^5} \\
&+ \cdots
\end{aligned}$$

etc. etc.

IV.

From the formula:

$$\frac{\left(\frac{2K}{\pi}\right)^2}{\sin^2 \operatorname{am} \frac{2Kx}{\pi}}$$
$$= \frac{2K}{\pi}\left(\frac{2K}{\pi} - \frac{2E^{\mathrm{I}}}{\pi}\right) + \frac{1}{\sin^2 x} - 4\left\{\frac{2q^2\cos 2x}{1-q^2} + \frac{4q^4\cos 4x}{1-q^4} + \frac{6q^6\cos 6x}{1-q^6} + \cdots\right\}$$

you obtain the following:

$$\begin{aligned}
&\frac{2\cdot 3\left(\frac{2K}{\pi}\right)^4}{\sin^4 \operatorname{am} \frac{2Kx}{\pi}} \\
&= 4(1+k^2)\left(\frac{2K}{\pi}\right)^3\left(\frac{2K}{\pi} - \frac{2E^{\mathrm{I}}}{\pi}\right) - 2k^2\left(\frac{2K}{\pi}\right)^4 \\
&+ \frac{4(1+k^2)\left(\frac{2K}{\pi}\right)^2}{\sin^2 x} + \frac{d^2}{dx^2}\frac{1}{\sin^2 x} \\
&- 4\left\{2\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 2^3\right\}\frac{q^2\cos 2x}{1-q^2} \\
&- 4\left\{4\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 4^3\right\}\frac{q^4\cos 4x}{1-q^4} \\
&- 4\left\{6\cdot 4(1+k^2)\left(\frac{2K}{\pi}\right)^2 - 6^3\right\}\frac{q^6\cos 6x}{1-q^6} \\
&- \cdots
\end{aligned}$$

$$
\begin{aligned}
&\frac{2\cdot 3\cdot 4\cdot 5\left(\frac{2K}{\pi}\right)^6}{\sin^6 \operatorname{am}\frac{2Kx}{\pi}}\\
&= 8(8+7k^2+8k^4)\left(\frac{2k}{\pi}\right)^5\left(\frac{2K}{\pi}-\frac{2E^{\mathrm{I}}}{\pi}\right)-32k^2(1+k^2)\left(\frac{2K}{\pi}\right)^6\\
&+\frac{8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4}{\sin^2 x}+20(1+k^2)\left(\frac{2K}{\pi}\right)^2\frac{d^2}{dx^2}\frac{1}{\sin^2 x}+\frac{d^4}{dx^4}\frac{1}{\sin^2 x}\\
&-4\left\{2\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4-2^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2+2^5\right\}\frac{q^2\cos 2x}{1-q^2}\\
&-4\left\{4\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4-4^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2+4^5\right\}\frac{q^4\cos 4x}{1-q^4}\\
&-4\left\{6\cdot 8(8+7k^2+8k^4)\left(\frac{2K}{\pi}\right)^4-6^3\cdot 20(1+k^2)\left(\frac{2K}{\pi}\right)^2+6^5\right\}\frac{q^6\cos 6x}{1-q^6}\\
&-\cdots
\end{aligned}
$$

etc. etc.

45.

The examples propounded in the preceding paragraphs tell us how the expansions of the functions $\sin^n \operatorname{am}\frac{2Kx}{\pi}$, $\frac{1}{\sin^n \operatorname{am}\frac{2Kx}{\pi}}$ are found from the formulas (2.), (3.), (6.), (7.) in § 43. The quantities $A_n^{(m)}$, $B_n^{(m)}$ on which they depend can be found successively by means of the formulas (4.), (5.) of the same paragraph. But to answer the question how to obtain general expressions for them, because they become too complicated to find them by induction, one has to elaborate a little more on this. For this purpose, we mention the following things in advance.

The following elementary formula is known:

$$\sin \operatorname{am}(u+v)-\sin \operatorname{am}(u-v)=\frac{2\sin \operatorname{am} v\cos \operatorname{am} u\Delta \operatorname{am} u}{1-k^2\sin^2 \operatorname{am} u\sin^2 \operatorname{am} v},$$

having integrated which with respect to u it results:

$$(1.)\qquad \int\limits_0^u du\,\{\sin\mathrm{am}(u+v)-\sin\mathrm{am}(u-v)\}=\frac{1}{k}\ln\left(\frac{1+k\sin\mathrm{am}\,u\sin\mathrm{am}\,v}{1-k\sin\mathrm{am}\,u\sin\mathrm{am}\,v}\right).$$

From Talyor's theorem it is:

$$\sin\mathrm{am}(u+v)-\sin\mathrm{am}(u-v)=2\left\{\frac{d\sin\mathrm{am}\,u}{du}\cdot v+\frac{d^3\sin\mathrm{am}\,u}{du^3}\cdot\frac{v^3}{\Pi 3}+\frac{d^5\sin\mathrm{am}\,u}{du^5}\cdot\frac{v^5}{\Pi 5}+\cdots\right\},$$

whence it follows:

$$\int\limits_0^u du\,\{\sin\mathrm{am}(u+v)-\sin\mathrm{am}(u-v)\}=2\left\{\sin\mathrm{am}\,u\cdot v+\frac{d^2\sin\mathrm{am}\,u}{du^2}\cdot\frac{v^3}{\Pi 3}+\frac{d^4\sin\mathrm{am}\,u}{du^4}\cdot\frac{v^5}{\Pi 5}+\cdots\right\}.$$

For, having put $u=0$ it is easily seen that both $\sin\mathrm{am}\,u$ and in general $\frac{d^{2m}\sin\mathrm{am}\,u}{du^{2m}}$ vanish. Hence equation (1.), having also expanded its other side, goes over into this one:

$$(2.)\qquad \sin\mathrm{am}\,u\cdot v+\frac{d^2\sin\mathrm{am}\,u}{du^2}\cdot\frac{v^3}{\Pi 3}+\frac{d^4\sin\mathrm{am}\,u}{du^4}\cdot\frac{v^5}{\Pi 5}+\text{etc.}$$

$$=\sin\mathrm{am}\,u\sin\mathrm{am}\,v+\frac{k^2}{3}\sin^3\mathrm{am}\,u\sin^3\mathrm{am}\,v+\frac{k^4}{5}\sin^5\mathrm{am}\,u\sin^5\mathrm{am}\,v+\cdots$$

Further, having multiplied the known equations:

$$\sin\mathrm{am}(u+v)+\sin\mathrm{am}(u-v)=\frac{2\sin\mathrm{am}\,u\cos\mathrm{am}\,v\Delta\,\mathrm{am}\,v}{1-k^2\sin^2\mathrm{am}\,u\sin^2\mathrm{am}\,v}$$
$$\sin\mathrm{am}(u+v)-\sin\mathrm{am}(u-v)=\frac{2\sin\mathrm{am}\,v\cos\mathrm{am}\,u\Delta\,\mathrm{am}\,u}{1-k^2\sin^2\mathrm{am}\,u\sin^2\mathrm{am}\,v}$$

by each other we obtain:

$$(2.)\qquad \sin^2\mathrm{am}(u+v)-\sin^2\mathrm{am}(u-v)$$
$$=\frac{4\sin\mathrm{am}\,u\cos\mathrm{am}\,u\Delta\,\mathrm{am}\,u\cdot\sin\mathrm{am}\,v\cos\mathrm{am}\,v\Delta\,\mathrm{am}\,v}{[1-k^2\sin^2\mathrm{am}\,u\sin^2\mathrm{am}\,v]^2}=\frac{d\sin^2\mathrm{am}\,u\cdot d\sin^2\mathrm{am}\,v}{[1-k^2\sin^2\mathrm{am}\,u\sin^2\mathrm{am}\,v]^2dudv}.$$

Having done the integration with respect to v this equation results:

$$\int_0^v dv \left\{\sin^2 \mathrm{am}(u+v) - \sin^2 \mathrm{am}(u-v)\right\}$$

$$= \frac{2\sin \mathrm{am}\, u \cos \mathrm{am}\, u \Delta \mathrm{am}\, u \cdot \sin^2 \mathrm{am}\, v}{1 - k^2 \sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, v} = \frac{\sin^2 \mathrm{am}\, v \cdot d \sin^2 \mathrm{am}\, u}{(1 - k^2 \sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, v) du}$$

Having integrated this equation once again but with respect to the other variable u we obtain:

$$(4.) \qquad \int_0^u du \int_0^v dv \left\{\sin^2 \mathrm{am}(u+v) - \sin^2 \mathrm{am}(u-v)\right\} = -\frac{1}{k^2} \ln(1 - k^2 \sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, v).$$

From Taylor's theorem it is:

$$\sin^2 \mathrm{am}(u+v) - \sin^2 \mathrm{am}(u-v)$$

$$2\left\{\frac{d \sin^2 \mathrm{am}\, u}{du} \cdot v + \frac{d^3 \sin^2 \mathrm{am}\, u}{du^3} \cdot \frac{v^3}{\Pi 3} + \frac{d^5 \sin^2 \mathrm{am}\, u}{du^5} \cdot \frac{v^5}{\Pi 5} + \cdots\right\},$$

whence:

$$\int_0^v dv \left\{\sin^2 \mathrm{am}(u+v) - \sin^2 \mathrm{am}(u-v)\right\}$$

$$2\left\{\frac{d \sin^2 \mathrm{am}\, u}{du} \cdot \frac{v^2}{\Pi 2} + \frac{d^3 \sin^2 \mathrm{am}\, u}{du^3} \cdot \frac{v^4}{\Pi 4} + \frac{d^5 \sin^2 \mathrm{am}\, u}{du^5} \cdot \frac{v^6}{\Pi 6} + \cdots\right\}$$

$$\int_0^u du \int_0^v dv \left\{\sin^2 \mathrm{am}(u+v) - \sin^2 \mathrm{am}(u-v)\right\}$$

$$2\left\{\sin^2 \mathrm{am}\, u \cdot \frac{v^2}{\Pi 2} + \frac{d^2 \sin^2 \mathrm{am}\, u}{du^2} \cdot \frac{v^4}{\Pi 4} + \frac{d^4 \sin^2 \mathrm{am}\, u}{du^4} \cdot \frac{v^6}{\Pi 6} + \cdots\right\} - 2\left\{U^{(2)} \frac{v^4}{\Pi 4} + U^{(4)} \frac{v^6}{\Pi 6} + \cdots\right\}$$

if we denote the value of the expression $\frac{d^{2m} \sin^2 \mathrm{am}\, u}{du^{2m}}$ which it obtains for $u = 0$ by the character $U^{(2m)}$. Hence equation (4.), having also expanded its other side, goes over into this one:

$$(5.)\quad \sin^2 \operatorname{am} u \cdot \frac{v^2}{\Pi 2} + \frac{d^2 \sin^2 \operatorname{am} u}{du^2} \cdot \frac{v^4}{\Pi 4} + \frac{d^4 \sin^2 \operatorname{am} u}{du^4} \cdot \frac{v^6}{\Pi 6} + \cdots - \left\{ U^{(2)} \frac{v^4}{\Pi 4} + U^{(4)} \frac{v^6}{\Pi 6} + \cdots \right\}$$

$$= \frac{1}{2} \sin^2 \operatorname{am} u \sin^2 \operatorname{am} v + \frac{k^2}{4} \sin^4 \operatorname{am} u \sin^4 \operatorname{am} v + \frac{k^4}{6} \sin^6 \operatorname{am} u \sin^6 \operatorname{am} v + \cdots$$

Having prepared these things put

$$u = \sin \operatorname{am} u + R_1 \sin^3 \operatorname{am} u + R_2 \sin^5 \operatorname{am} u + +R_3 \sin^7 \operatorname{am} u + \cdots,$$

and in general

$$\begin{aligned} u^n &= [\sin \operatorname{am} u + R_1 \sin^3 \operatorname{am} u + R_2 \sin^5 \operatorname{am} u + +R_3 \sin^7 \operatorname{am} u + \cdots]^n \\ &= \sin^n \operatorname{am} u + R_1^{(n)} \sin^{n+2} \operatorname{am} u + R_3^{(n)} \sin^{n+4} \operatorname{am} u + R_5^{(n)} \sin^{n+6} \operatorname{am} u + \cdots; \end{aligned}$$

further, from the inversion of the series:

$$\sin \operatorname{am} u + R_1 \sin^3 \operatorname{am} u + R_2 \sin^5 \operatorname{am} u + +R_3 \sin^7 \operatorname{am} u + \cdots$$

let this one result:

$$\sin \operatorname{am} u = u + S_1 u^3 + S_2 u^5 + S_3 u^7 + \cdots,$$

and let it again be:

$$\sin^n \operatorname{am} u = [u + S_1 u^3 + S_2 u^5 + S_3 u^7 + \cdots]^n = u^n + S_1^{(n)} u^{n+2} + S_2^{(n)} u^{n+4} + S_3^{(n)} u^{n+6} + \cdots,$$

Now, from equation (2.):

$$\begin{aligned} &\sin \sin \operatorname{am} u \cdot v + \frac{d^2 \sin \operatorname{am} u}{du^2} \cdot \frac{v^3}{\Pi 3} + \frac{d^4 \sin \operatorname{am} u}{du^4} \cdot \frac{v^5}{\Pi 5} + \cdots \\ &= \sin \operatorname{am} u \sin \operatorname{am} v + \frac{k^2}{3} \sin^3 \operatorname{am} u \sin^3 \operatorname{am} v + \frac{k^4}{5} \sin^5 \operatorname{am} u \sin^5 \operatorname{am} v + \cdots, \end{aligned}$$

having expanded v, v^3, v^5 etc. into a series of powers of $\sin \operatorname{am} v$ and having compared the coefficients of $\sin^{2n+1} \operatorname{am} v$ on both sides of the equation this equation results:

$$(6.)\qquad \frac{k^{2n}\sin^{2n+1}\mathrm{am}}{2n+1}$$

$$= R_n^{(1)}\sin\mathrm{am} + R_{n-1}^{(3)}\frac{d^2\sin\mathrm{am}}{\Pi 3\cdot du^2} + R_{n-2}^{(5)}\frac{d^4\sin\mathrm{am}}{\Pi 5\cdot du^4} + \cdots \frac{d^{2n}\sin\mathrm{am}\,u}{\Pi(2n+1)du^{2n}}.$$

In the same way from formula (5.) this equation results:

$$(7.)\qquad \frac{k^{2n-2}\sin^{2n}\mathrm{am}}{2n}$$

$$= R_{n-1}^{(2)}\frac{\sin^2\mathrm{am}\,u}{\Pi 2} + R_{n-2}^{(4)}\frac{d^2\sin^2\mathrm{am}\,u}{\Pi 4\cdot du^2} + R_{n-3}^{(6)}\frac{d^4\sin^2\mathrm{am}\,u}{\Pi 6\cdot du^4} + \cdots + \frac{d^{2n-2}\sin^2\mathrm{am}\,u}{\Pi(2n)\cdot du^{2n-2}}$$

$$-\left\{\frac{R_{n-2}^{(4)}}{3\cdot 4} + \frac{R_{n-3}^{(6)}}{5\cdot 6}S_1^{(2)} + \frac{R_{n-4}^{(8)}}{7\cdot 8}S_2^{(2)} + \cdots + \frac{S_{n-2}^{(2)}}{(2n-1)\cdot 2n}\right\}$$

From (6.), (7.) having changed u into $u + iK'$ it follows:

$$(8.)\qquad \frac{1}{(2n+1)\sin^{2n+1}\mathrm{am}\,u}$$

$$= \frac{R_n^{(1)}}{\sin\mathrm{am}\,u} + \frac{R_{n-1}^{(3)}}{\Pi 3}\cdot\frac{d^2}{du^2}\frac{1}{\sin\mathrm{am}\,u} + \frac{R_{n-2}^{(5)}}{\Pi 5}\cdot\frac{d^4}{du^4}\frac{1}{\sin\mathrm{am}\,u} + \cdots + \frac{1}{\Pi(2n+1)}\cdot\frac{d^{2n}}{du^{2n}}\frac{1}{\sin\mathrm{am}\,u}$$

$$(9.)\qquad \frac{1}{(2n)\sin^{2n}\mathrm{am}\,u}$$

$$= \frac{R_{n-1}^{(2)}}{\Pi 2\cdot\sin^2\mathrm{am}\,u} + \frac{R_{n-2}^{(4)}}{\Pi 4}\cdot\frac{d^2}{du^2}\frac{1}{\sin^2\mathrm{am}\,u} + \frac{R_{n-3}^{(6)}}{\Pi 6}\cdot\frac{d^4}{du^4}\frac{1}{\sin^2\mathrm{am}\,u} + \cdots + \frac{1}{\Pi(2n)}\cdot\frac{d^{2n-2}}{du^{2n-2}}\frac{1}{\sin^2\mathrm{am}\,u}$$

$$-k^2\left\{\frac{R_{n-2}^{(4)}}{3\cdot 4} + \frac{R_{n-3}^{(6)}}{5\cdot 6}S_1^{(2)} + \frac{R_{n-4}^{(8)}}{7\cdot 8}S_2^{(2)} + \cdots + \frac{S_{n-2}^{(2)}}{(2n-1)2n}\right\}.$$

These are the general formulas we are looking for by means of which $\sin^n\mathrm{am}\,u$, $\frac{1}{\sin^n\mathrm{am}\,u}$ are found from $\sin\mathrm{am}\,u$, $\sin^2\mathrm{am}\,u$, $\frac{1}{\sin\mathrm{am}\,u}$, $\frac{1}{\sin^2\mathrm{am}\,u}$ and its differentials.

On this occasion I remark, if vice versa $\sin\mathrm{am}\,v$, $\sin^2\mathrm{am}\,v$, $\sin^3\mathrm{am}\,v$, etc. are expanded into a power series in v, that from the formulas (2.), (5.) it is found:

$$(10.)\qquad \frac{d^{2n}\sin\mathrm{am}\,u}{\Pi(2n+1)du^{2n+1}}$$

$$= S_n^{(1)} \sin \operatorname{am} u + \frac{k^2}{3} S_{n-1}^{(3)} \sin^3 \operatorname{am} u + \frac{k^4}{5} S_{n-2}^{(5)} \sin^5 \operatorname{am} + \cdots \frac{k^{2n}}{2n+1} \sin^{2n+1} \sin^{2n+1} \operatorname{am} u$$

$$(11.) \qquad \frac{d^{2n} \sin^2 \operatorname{am} u}{\Pi(2n+2) du^{2n}} - \frac{S_{n-1}^{(2)}}{(2n+1)(2n+2)}$$

$$= \frac{1}{2} S_n^{(2)} \sin^2 \operatorname{am} u + \frac{k^2}{4} S_{n-1}^{(4)} \sin^4 \operatorname{am} u + \frac{k^3}{6} S_{n-2}^{(6)} \sin^6 \operatorname{am} u + \cdots + \frac{k^{2n}}{2n+2} \sin^{2n+2} \operatorname{am} u.$$

Finally, some things concerning the invention of $R_m^{(n)}$, $S_m^{(n)}$ are to be added. Having put $\sin \operatorname{am} u = y$, from the propounded definition it is:

$$u = \int_0^y \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = y + R_1 y^3 + R_2 y^5 + R_3 y^7 + \cdots$$

or:

$$\frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = 1 + 3R_1 y^2 + 5R_2 y^4 + 7R_3 y^6 + \cdots ;$$

hence it is:

$$\begin{aligned}
&3R_1 = \frac{1+k^2}{2}, \quad 5R_2 = \frac{1 \cdot 3}{2 \cdot 4} + \frac{1}{2} \cdot \frac{1}{2} k^2 + \frac{1 \cdot 3}{2 \cdot 4} k^4 \\
&7R_3 = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{2} k^2 + \frac{1}{2} \cdot \frac{1 \cdot 3}{2 \cdot 4} k^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} k^6 \\
&9R_4 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{2} k^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 3}{2 \cdot 4} k^4 + \frac{1}{2} \cdot \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} k^6 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{\cdot 2 \cdot 4 \cdot 6 \cdot 8} k^8 \\
&\qquad\qquad \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}$$

or also:

$$
\begin{aligned}
3R_1 &= \frac{1}{2} \cdot (1+k^2) \\
5R_2 &= \frac{1\cdot 3}{2\cdot 4} \cdot (1+k^2)^2 - \frac{1}{2} \cdot k^2 \\
7R_3 &= \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} \cdot (1+k^2)^3 - \frac{1\cdot 3}{2\cdot 2} \cdot k^2(1+k^2) \\
9R_4 &= \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8} \cdot (1+k^2)^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 2} \cdot k^2(1+k^2)^2 + \frac{1\cdot 3}{2\cdot 4}k^4 \\
11R_5 &= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 10} \cdot (1+k^2)^5 - \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 2} \cdot k^2(1+k^2)^3 + \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 4}k^4(1+k^2) \\
13R_6 &= \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12} \cdot (1+k^2)^6 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8\cdot 2} \cdot k^2(1+k^2)^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 2\cdot 4}k^4(1+k^2)^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}k^6
\end{aligned}
$$

or also:

$$
\begin{aligned}
3R_1 &= 1 - \frac{1}{2}\cdot 1\cdot k'^2 \\
5R_2 &= 1 - \frac{1}{2}\cdot 2\cdot k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 1\cdot k'^4 \\
7R_3 &= 1 - \frac{1}{2}\cdot 3\cdot k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 3\cdot k'^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot 1\cdot k'^6 \\
9R_4 &= 1 - \frac{1}{2}\cdot 4\cdot k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 6\cdot k'^4 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot 4\cdot k'^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}k'^8 \\
&\qquad\qquad \text{etc.} \qquad\qquad \text{etc.}
\end{aligned}
$$

or finally:

$$
\begin{aligned}
3R_1 &= k^2 + \frac{1}{2}\cdot k'^2 \\
5R_2 &= k^4 + \frac{1}{2}\cdot 2k^2k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot k'^4 \\
7R_3 &= k^6 + \frac{1}{2}\cdot 3k^4k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 3k^2\cdot k'^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot k'^6 \\
9R_4 &= k^8 + \frac{1}{2}\cdot 4k^6k'^2 + \frac{1\cdot 3}{2\cdot 4}\cdot 6k^2\cdot k'^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot 4k^2k'^6 + \frac{1\cdot 3\cdot 7\cdot 9}{2\cdot 4\cdot 6\cdot 8}k'^8 \\
&\qquad \text{etc.} \qquad\qquad\qquad \text{etc.}
\end{aligned}
$$

From these four ways to express the quantities R_m the second way yields a sufficiently memorable and beautiful representation of them, if we introduce the quantities:

$$r = \frac{1+k^2}{2k}.$$

For example it is:

$$\frac{13R_6}{k^6} = \frac{1\cdot 3\cdots 11}{1\cdot 2\cdots 6}r^6 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 2}r^4 + \frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 2\cdot 4}r^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6},$$

having integrated which expression 6 times with respect to r we obtain:

$$13\int^6 \frac{R_6 dr^6}{k^6} = \frac{r^{12}}{2\cdot 4\cdots 12} - \frac{r^{10}}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 2} + \frac{r^8}{2\cdot 4\cdot 6\cdot 8\cdot 2\cdot 4} - \frac{r^6}{2\cdot 4\cdot 6\cdot 2\cdot 4\cdot 6} + C'r^4 + C''r^2 + C''',$$

C', C'', C''' denoting arbitrary constants. Having determined them this equation results:

$$13\int^6 \frac{R_6 dr^6}{k^6} = \frac{(r^2-1)^6}{2^6\cdot \Pi 6},$$

whence vice versa it is:

$$13R_6 = \frac{k^6 d(r^2-1)^6}{2^6\cdot \Pi 6\cdot dr^6};$$

and in the same way it is obtained in general:

$$(12.) \qquad (2m+1)R_m = \frac{k^m d^m (r^2-1)^m}{2^m\cdot \Pi m\cdot dr^m}.$$

Confer the short commentary (*Crelle Journal II. p.223*) with the title:

"Ueber eine besondere Gattungen algebraischer Functionen, die aus der Entwicklung der Function $(1-2xz+z^2)^{-\frac{1}{2}}$ entstehn."

Having found the quantities R_m by means of known algorithms one has to find quantities $R_m^{(n)}$, $S_m^{(n)}$ that it is:

$$[1 + R_1 x + R_2 x^2 + R_3 x^3 + \cdots]^n = 1 + R_1^{(n)} x + R_2^{(n)} x^2 + R_3^{(n)} x^3 + \cdots,$$

further, if it is put:

$$y = x[1 + R_1 x^2 + R_2 x^4 + R_3 x^6 + \cdots],$$

let:

$$x^n = y^n[1 + S_1^{(n)} y^2 + s_2^{(n)} y^4 + S_3^{(n)} y^6 + \cdots];$$

these agree with the definition of the quantities $R_m^{(n)}$, $S_m^{(n)}$ propounded above. But having put:

$$\varphi(x) = 1 + R_1 x + R_2 x^2 + R_3 x^3 + \cdots,$$

from a theorem found by MacLaurin and Lagrange it is:

$$\begin{aligned} R_m^{(n)} &= \frac{d^m[\varphi(x)]^n}{\Pi m \cdot dx^m} \\ S_m^{(n)} &= \frac{n}{2m+n} \cdot \frac{d^m[\varphi(x)]^{-(2m+n)}}{\Pi m \cdot dx^m}, \end{aligned}$$

if one puts $x = 0$ after the differentiation.

46.

By means of the formulas (6.), (7.), (8.), (9.) of § 45 we obtain the following general expansions:

$$(1.) \qquad \frac{\left(\frac{2kK}{\pi}\right)^{2n+1} \sin^{2n+1} \operatorname{am} \frac{2Kx}{\pi}}{2n+1}$$

$$\begin{aligned} &= 4\left\{ R_n^{(1)} \left(\frac{2K}{\pi}\right)^{2n} - \frac{R_{n-1}^{(3)}}{\Pi 3} \left(\frac{2K}{\pi}\right)^{2n-2} + \frac{R_{n-2}^{(5)}}{\Pi 5} \left(\frac{2K}{\pi}\right)^{2n-4} - \cdots + \frac{(-1)^n}{\Pi(2n+1)} \right\} \frac{\sqrt{q} \sin x}{1-q} \\ &+ 4\left\{ R_n^{(1)} \left(\frac{2K}{\pi}\right)^{2n} - \frac{3^2 R_{n-1}^{(3)}}{\Pi 3} \left(\frac{2K}{\pi}\right)^{2n-2} + \frac{3^4 R_{n-2}^{(5)}}{\Pi 5} \left(\frac{2K}{\pi}\right)^{2n-4} - \cdots + \frac{(-1)^n 3^{2n}}{\Pi(2n+1)} \right\} \frac{\sqrt{q^3} \sin 3x}{1-q^3} \\ &= 4\left\{ R_n^{(1)} \left(\frac{2K}{\pi}\right)^{2n} - \frac{5^2 R_{n-1}^{(3)}}{\Pi 3} \left(\frac{2K}{\pi}\right)^{2n-2} + \frac{5^4 R_{n-2}^{(5)}}{\Pi 5} \left(\frac{2K}{\pi}\right)^{2n-4} - \cdots + \frac{(-1)^n 5^{2n}}{\Pi(2n+1)} \right\} \frac{\sqrt{q^5} \sin 5x}{1-q^5} \\ &+ \cdots \end{aligned}$$

$$(2.)\qquad \frac{\left(\frac{2kK}{\pi}\right)^{2n}\sin^{2n}\operatorname{am}\frac{2Kx}{\pi}}{2n}$$

$$= \frac{R^{(2)}_{n-1}}{\Pi 2}\left(\frac{2K}{\pi}\right)^{2n-1}\left(\frac{2K}{\pi}-\frac{2E^{\mathrm{I}}}{\pi}\right) - k^2\left(\frac{2K}{\pi}\right)^{2n}\left\{\frac{R^{(4)}_{n-2}}{3\cdot 4}+\frac{R^{(6)}_{n-3}}{5\cdot 6}S^{(2)}_1+\frac{R^{(8)}_{n-4}}{7\cdot 8}S^{(2)}_2+\cdots+\frac{S^{(2)}_{n-2}}{(2n-1)2n}\right\}$$

$$-4\left\{\frac{2R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{2^3R^{(4)}_{n-3}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\cdots+\frac{(-1)^{n-1}2^{2n-1}}{\Pi 2n}\right\}\frac{q\cos 2x}{1-q^2}$$

$$-4\left\{\frac{4R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{4^3R^{(4)}_{n-3}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\cdots+\frac{(-1)^{n-1}4^{2n-1}}{\Pi 2n}\right\}\frac{q^2\cos 4x}{1-q^4}$$

$$-4\left\{\frac{6R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{6^3R^{(4)}_{n-3}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\cdots+\frac{(-1)^{n-1}6^{2n-1}}{\Pi 2n}\right\}\frac{q^3\cos 6x}{1-q^6}$$

$$-\cdots$$

$$(3.)\qquad \frac{\left(\frac{2K}{\pi}\right)^{2n+1}}{(2n+1)\sin^{2n+1}\operatorname{am}\frac{2Kx}{\pi}}$$

$$= \frac{R^{(1)}_n\left(\frac{2K}{\pi}\right)^{2n}}{\sin x}+\frac{R^{(3)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 3}\cdot\frac{d^2}{dx^2}\frac{1}{\sin x}+\cdots+\frac{1}{\Pi(2n+1)}\cdot\frac{d^{2n}}{dx^{2n}}\frac{1}{\sin x}$$

$$+4\left\{R^{(1)}_n\left(\frac{2K}{\pi}\right)^{2n}-\frac{R^{(3)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 3}+\cdots+\frac{(-1)^n}{\Pi(2n+1)}\right\}\frac{q\sin x}{1-q}$$

$$+4\left\{R^{(1)}_n\left(\frac{2K}{\pi}\right)^{2n}-\frac{3^2R^{(3)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 3}+\cdots+\frac{(-1)^n3^{2n}}{\Pi(2n+1)}\right\}\frac{q^3\sin 3x}{1-q^3}$$

$$+4\left\{R^{(1)}_n\left(\frac{2K}{\pi}\right)^{2n}-\frac{5^2R^{(3)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 3}+\cdots+\frac{(-1)^n5^{2n}}{\Pi(2n+1)}\right\}\frac{q^5\sin 5x}{1-q^5}$$

$$+\cdots$$

$$(4.)\qquad \frac{\left(\frac{2K}{\pi}\right)^{2n}}{2n\cdot\sin^{2n}\operatorname{am}\frac{2Kx}{\pi}}$$

$$= \frac{1}{2}R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-1}\left(\frac{2K}{\pi}-\frac{2E^{\mathrm{I}}}{\pi}\right) - k^2\left(\frac{2K}{\pi}\right)^{2n}\left\{\frac{1}{3\cdot 4}R^{(4)}_{n-2}+\frac{1}{5\cdot 6}R^{(6)}_{n-3}S^{(2)}_1+\frac{1}{7\cdot 8}R^{(8)}_{n-4}S^{(2)}_2+\dots+\frac{1}{(2n-1)2n}S^{(2)}_{n-2}\right\}$$

$$+\frac{R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}+\frac{R^{(4)}_{n-2}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}\cdot\frac{d^2}{dx^2}\frac{1}{\sin^2 x}+\dots+\frac{1}{\Pi 2n}\cdot\frac{d^{2n-2}}{dx^{2n-2}}\frac{1}{\sin^2 x}$$

$$-4\left\{\frac{2R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{2^3R^{(4)}_{n-2}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\dots+\frac{(-1)^n 2^{2n-1}}{\Pi 2n}\right\}\frac{q^2\cos 2x}{1-q^2}$$

$$-4\left\{\frac{2R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{4^3R^{(4)}_{n-2}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\dots+\frac{(-1)^n 4^{2n-1}}{\Pi 2n}\right\}\frac{q^4\cos 4x}{1-q^4}$$

$$-4\left\{\frac{2R^{(2)}_{n-1}\left(\frac{2K}{\pi}\right)^{2n-2}}{\Pi 2}-\frac{6^3R^{(4)}_{n-2}\left(\frac{2K}{\pi}\right)^{2n-4}}{\Pi 4}+\dots+\frac{(-1)^n 6^{2n-1}}{\Pi 2n}\right\}\frac{q^6\cos 6x}{1-q^6}$$

$$-\dots$$

From the formulas (6.), (7.), (8.), (9.) of § 45 one can deduce others involving the functions $\cos\operatorname{am}u$, $\tan\operatorname{am}u$, $\Delta\operatorname{am}u$ instead of $\sin\operatorname{am}u$. For, from the formula:

$$\sin\operatorname{am}\left(k'u,\frac{ik}{k'}\right)=\cos\operatorname{coam}u,$$

whence it also is:

$$\sin\operatorname{am}\left(k'(K-u),\frac{ik}{k'}\right)=\cos\operatorname{am}u,$$

we see that in the propounded formulas, if one puts $\frac{ik}{k'}$ instead of k and $k'(K-u)$ instead of u, $\sin\operatorname{am}u$ goes over into $\cos\operatorname{am}u$, whence one finds similar formulas corresponding to $\cos\operatorname{am}u$. Further, from the equation:

$$\sin\operatorname{am}iu = i\tan\operatorname{am}(u,k')$$

it is clear that at the same time one can change u into iu, k into k', $\sin\operatorname{am}u$ into $i\tan\operatorname{am}u$; hence we find formulas for $\tan\operatorname{am}u$. Finally, from these, because

$$\cot\operatorname{am}(u+iK')=-i\Delta\operatorname{am}u,$$

one can find formulas for $\Delta\,\mathrm{am}\,u$ corresponding to the formulas (6.), (7.), (8.), (9.). Having found these by means of a similar method from the expansion of the functions:

$$\begin{array}{llll}
\cos\mathrm{am}\,\dfrac{2Kx}{\pi}, & \cos^2\mathrm{am}\,\dfrac{2Kx}{\pi}, & \Delta\,\mathrm{am}\,\dfrac{3Kx}{\pi}, & \Delta^2\,\mathrm{am}\,\dfrac{2Kx}{\pi} \\
\dfrac{1}{\cos\mathrm{am}\,\frac{2Kx}{\pi}}, & \dfrac{1}{\cos^2\mathrm{am}\,\frac{2Kx}{\pi}}, & \dfrac{1}{\Delta\,\mathrm{am}\,\frac{2Kx}{\pi}}, & \dfrac{1}{\Delta^2\,\mathrm{am}\,\frac{2Kx}{\pi}}
\end{array}$$

propounded by us one deduces general expansions of the functions:

$$\cos^n\mathrm{am}\,\frac{2Kx}{\pi}, \qquad \Delta^n\,\mathrm{am}\,\frac{2Kx}{\pi}.$$

It shall be sufficient to have mentioned these things.

We obtain extraordinary transformations of the series into which we expanded the elliptic functions after having put ix instead of x and applied the formulas we gave for the reduction of an imaginary argument to a real argument in the first foundations. But because those are easy to obtain we do not want to treat this subject here any longer.

2.3 The second kind of elliptic Functions is expanded into series

47.

Having integrated the integral formula exhibited above in § 41 (1.):

$$\left(\frac{2kK}{\pi}\right)^2\sin^2\mathrm{am}\,\frac{2Kx}{\pi} = \frac{2K}{\pi}\frac{2K}{\pi} - 4\left\{\frac{2q\cos 2x}{1-q^2} + \frac{4q^2\cos 4x}{1-q^4} + \frac{6q^3\cos 6x}{1-q^6} + \cdots\right\}$$

from $x = 0$ to $x = x$ it results:

$$\left(\frac{2kK}{\pi}\right)^2\int\limits_0^x \sin^2\mathrm{am}\,\frac{2Kx}{\pi}$$

$$= \left\{\frac{2K}{\pi}\frac{2k}{\pi} - \frac{2K}{\pi}\frac{2E^{\mathrm{I}}}{\pi}\right\}x - 4\left\{\frac{q\sin 2x}{1-q^2} + \frac{q\sin 4x}{1-q^4} + \frac{q^3\sin 6x}{1-q^6} + \frac{q^4\sin 8x}{1-q^8} + \cdots\right\}.$$

In the following, let us denote the following expression by the character $\frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right)$:

$$(1.)\qquad \frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right)=\frac{2Kx}{\pi}\left(\frac{2K}{\pi}-\frac{2E^{\mathrm{I}}}{\pi}\right)-\left(\frac{2kK}{\pi}\right)^2\int\limits_0^x \sin^2 \operatorname{am}\frac{2Kx}{\pi}dx$$
$$=4\left\{\frac{q\sin 2x}{1-q^2}+\frac{q^2\sin 4x}{1-q^4}+\frac{q^3\sin 6x}{1-q^6}+\frac{q^4\sin 8x}{1-q^8}+\cdots\right\}.$$

From Legendre's notation having put $\frac{2Kx}{\pi}=u$, $\varphi=\operatorname{am}u$ it will be:

$$(2.)\qquad Z(u)=\frac{F^{\mathrm{I}}E(\varphi)-E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}}.$$

It is convenient to introduce the function $Z(u)$ instead of $E(\varphi)$ into the analysis of elliptic functions; moreover, it is easy to reduce it to the functions used by Legendre by means of formula (2.). We want to sketch a little bit, how from the expansion of the function Z which formula (1.) yields it is possible to derive many of its properties, even though they are already known.

In (1.) change x to $x+\frac{\pi}{2}$, then it results:

$$\frac{2K}{\pi}Z\left(\frac{2kx}{\pi}+K\right)=-4\left\{\frac{q\sin 2x}{1-q^2}-\frac{q^2\sin 4x}{1-q^4}+\frac{q^3\sin 6x}{1-q^6}-\cdots\right\},$$

whence it is:

$$\frac{2k}{\pi}Z\left(\frac{2Kx}{\pi}\right)-\frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}+K\right)=8\left\{\frac{q\sin 2x}{1-q^2}+\frac{q^3\sin 6x}{1-q^6}+\frac{q^5\sin 10x}{1-q^{10}}+\cdots\right\}.$$

Further, in (1.) change x to $2x$, q to q^2, and at the same time k to $k^{(2)}$, K to $K^{(2)}$, then this equation results:

$$\frac{2K^{(2)}}{\pi}Z\left(\frac{4K^{(2)}}{\pi},k^{(2)}\right)=4\left\{\frac{q^2\sin 4x}{1-q^4}+\frac{q^4\sin 8x}{1-q^8}+\frac{q^6\sin 12x}{1-q^{12}}+\cdots\right\},$$

whence it is:

$$\frac{2K}{\pi} Z\left(\frac{2kx}{\pi}\right) - \frac{2K^{(2)}}{\pi} Z\left(\frac{4K^{(2)}}{\pi}, k^{(2)}\right) = \left\{\frac{q\sin 2x}{1-q^2} + \frac{q^3\sin 6x}{1-q^6} + \frac{q^5\sin 10x}{1-q^{10}} + \cdots\right\}.$$

But above we found:

$$\frac{2kK}{\pi}\sin\mathrm{am}\,\frac{2kX}{\pi} = \left\{\frac{\sqrt{q}\sin x}{1-q} + \frac{\sqrt{q^3}\sin 3x}{1-q^3} + \frac{\sqrt{q^5}\sin 5x}{1-q^5} + \cdots\right\},$$

whence having changed q to q^2, x to $2x$ it is:

$$\frac{2k^{(2)}K^{(2)}}{\pi}\sin\mathrm{am}\left(\frac{4K^{(2)}x}{\pi}, k^{(2)}\right) = 4\left\{\frac{q\sin 2x}{1-q^2} + \frac{q^3\sin 6x}{1-q^6} + \frac{q^5\sin 10x}{1-q^{10}} + \cdots\right\}.$$

Hence it follows:

$$(3.)\quad \frac{2K}{\pi}\left\{Z\left(\frac{2Kx}{\pi}\right) - Z\left(\frac{2Kx}{\pi} + K\right)\right\} = \frac{4k^{(2)}K^{(2)}}{\pi}\sin\mathrm{am}\left(\frac{4K^{(2)}x}{\pi}, k^{(2)}\right)$$

$$(4.)\quad \frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right) - \frac{2K^{(2)}}{\pi}Z\left(\frac{4K^{(2)}x}{\pi}, k^{(2)}\right) = \frac{2k^{(2)}K^{(2)}}{\pi}\sin\mathrm{am}\left(\frac{4K^{(2)}x}{\pi}, k^{(2)}\right)$$

$$(5.)\quad \frac{2K}{\pi}Z\left(\frac{2Kx}{\pi}\right) + \frac{2K}{\pi}Z\left(\frac{2Kx}{\pi} + K\right) - \frac{4K^{(2)}}{\pi}\left(\frac{4K^{(2)}x}{\pi}, k^{(2)}\right) = 0.$$

In these formulas, of which (4.), (5.) yield the transformation of the function Z of second order, it is:

$$k^{(2)} = \frac{1-k'}{1+k'},\quad K^{(2)} = \frac{1+k'}{2}\cdot K,\quad \sin\mathrm{am}\left(\frac{4K^{(2)}x}{\pi}, k^{(2)}\right) = (1+k')\sin\mathrm{am}\,\frac{2Kx}{\pi}\cdot\sin\mathrm{coam}\,\frac{2Kx}{\pi},$$

as it is known for the transformation of second order propounded by Legendre. Hence formula (3.) having put $u = \frac{2Kx}{\pi}$ can also be represented this way:

$$(6.)\qquad Z(u) - Z(u+K) = k^2\sin\mathrm{am}\,u\sin\mathrm{coam}\,u.$$

For the sake of brevity let us put $\operatorname{am}\left(\frac{2mK^{(m)}x}{\pi}, k^{(m)}\right) = \varphi^{(m)}$, from formula (4.), having successively put $k^{(2)}$,$k^{(4)}$, $k^{(8)}$, $k^{(16)} \cdots$ instead of k and $2x$, $4x$, $8x \cdots$, instead of x, this equation results:

$$(7.) \quad K \cdot Z(u) = F^{\mathrm{I}}E(\varphi) - E^{\mathrm{I}}F(\varphi) = k^{(2)}K^{(2)} \sin\varphi^{(2)} + k^{(4)}K^{(4)} \sin\varphi^{(4)} + k^{(8)}K^{(8)} \sin\varphi^{(8)} + \cdots,$$

which formula was given by Legendre.

In like manner from formula § 41:

$$\frac{2K}{\pi}\frac{2K}{\pi} - \frac{2K}{\pi}\frac{2E^{\mathrm{I}}}{\pi} = 8\left\{\frac{q}{(1-q)^2} + \frac{q^3}{(1-q^3)^2} + \frac{q^5}{(1-q^5)^2} + \frac{q^7}{(1-q^7)^2} + \cdots\right\},$$

which can also be expressed this way:

$$\frac{2K}{\pi}\frac{2K}{\pi} - \frac{2K}{\pi}\frac{2E^{\mathrm{I}}}{\pi} = 8\left\{\frac{q}{1-q^2} + \frac{2q^2}{1-q^4} + \frac{3q^3}{1-q^6} + \frac{4q^4}{1-q^8} + \cdots\right\},$$

having compared it to this one we found above:

$$\left(\frac{2kK}{\pi}\right)^2 = 16\left\{\frac{q}{1-q^2} + \frac{3q^3}{1-q^5} + \frac{5q^5}{1-q^{10}} + \frac{7q^7}{1-q^{14}} + \cdots\right\},$$

this equation results:

$$(8.) \qquad 2K(K - E^{\mathrm{I}}) = (kK)^2 + 2(k^{(2)}K^{(2)})^2 + 4(k^{(4)}K^{(4)})^2 + 8(k^{(8)}K^{(8)})^2 + \cdots,$$

which agrees with that one Gauss gave in his paper *Determinatio attractionis* etc. § 17.

48.

By means of the same method we used in § 41 to find the expansion of the expression $\left(\frac{2kK}{\pi}\right)^2 \sin^2 \operatorname{am} \frac{2Kx}{\pi}$, let us investigate, how to expand the expression $\left\{\frac{2K}{\pi} Z\left(\frac{2Kx}{\pi}\right)\right\}^2$ into a series. Let us prove:

$$\left(\frac{2K}{\pi}\right)^2 Z\left(\frac{2Kx}{\pi}\right) Z\left(\frac{2Kx}{\pi}\right) = 16\left\{\frac{q\sin 2x}{1-q^2} + \frac{q^2\sin 4x}{1-q^4} + \frac{q^3\sin 6x}{1-q^6} + \cdots\right\}^2$$
$$= 8\left\{A + A'\cos 2x + A''\cos 4x + A'''\cos 6x + \cdots\right\},$$

which expression we see to take on the propounded form, if one puts $\cos 2(m-m')x - \cos 2(m+m')x$ instead of $2\sin 2mx \sin m'x$ everywhere. At first it is:

$$A = \frac{q^2}{(1-q^2)^2} + \frac{q^4}{(1-q^4)^2} + \frac{q^6}{(1-q^6)^2} + \frac{q^8}{(1-q^8)^2} + \cdots$$

After this, in general we obtain: $A^{(n)} = 2B^{(n)} - C^{(n)}$, if it is put:

$$B^{(n)} = \frac{q^{n+2}}{(1-q^2)(1-q^{2n+2})} + \frac{q^{n+4}}{(1-q^4)(1-q^{2n+4})} + \frac{q^{n+6}}{(1-q^6)(1-q^{2n+6})} + \cdots$$
$$C^{(n)} = \frac{q^n}{(1-q^2)(1-q^{2n-2})} + \frac{q^n}{(1-q^4)(1-q^{2n-4})} + \cdots + \frac{q^n}{(1-q^{2n-2})(1-q^2)} + \cdots$$

In the single terms of these expressions respectively put:

$$\frac{q^{m+n}}{(1-q^m)(1-q^{2n+m})} = \frac{q^n}{1-q^{2n}}\left\{\frac{q^m}{1-q^m} - \frac{q^{2n+m}}{1-q^{2n+m}}\right\}$$
$$\frac{q^n}{(1-q^n)(1-q^{2n-m})} = \frac{q^n}{1-q^{2n}}\left\{\frac{q^m}{1-q^m} + \frac{q^{2n-m}}{1-q^{2n-m}} + 1\right\},$$

then it results:

$$B^{(n)} = \frac{q^n}{1-q^{2n}}\left\{\frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} + \frac{q^6}{1-q^6} + \cdots\right\}$$
$$- \frac{q^n}{1-q^{2n}}\left\{\frac{q^{2n+2}}{1-q^{2n+2}} + \frac{q^{2n+4}}{1-q^{2n+4}} + \frac{q^{2n+6}}{1-q^{2n+6}} + \cdots\right\}$$
$$= \frac{q^n}{1-q^{2n}}\left\{\frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} + \frac{q^6}{1-q^6} + \cdots + \frac{q^{2n}}{1-q^{2n}}\right\}$$
$$C^{(n)} = \frac{(n-1)q^n}{1-q^{2n}} + \frac{2q^n}{1-q^{2n}}\left\{\frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} + \frac{q^6}{1-q^6} + \cdots + \frac{q^{2n-2}}{1-q^{2n-2}}\right\};$$

hence:

$$A^{(n)} = 2B^{(n)} - C^{(n)} = -\frac{(n-1)q^n}{1-q^{2n}} + \frac{2q^{3n}}{(1-q^{2n})^2} = -\frac{nq^n}{1-q^{2n}} + \frac{q^n(1+q^{2n})}{(1-q^{2n})^2};$$

Having collected all these, one finds the expansion in question:

$$(1.)\qquad \left(\frac{2K}{\pi}\right)^2 Z\left(\frac{2Kx}{\pi}\right) Z\left(\frac{2Kx}{\pi}\right) = 8A - 8\left\{\frac{q\cos 2x}{1-q^2} + \frac{2q^2\cos 4x}{1-q^4} + \frac{3q^3\cos 6x}{1-q^6} + \cdots\right\}$$
$$+\,8\left\{\frac{q(1+q^2)\cos 2x}{(1-q^2)^2} + \frac{q^2(1+q^4)\cos 4x}{(1-q^4)^2} + \frac{q^3(1+q^6)\cos 6x}{(1-q^6)^2} + \cdots\right\}.$$

Because $A = \frac{q^2}{(1-q^2)^2} + \frac{q^4}{(1-q^4)^2} + \frac{q^6}{(1-q^6)^2} + \cdots$ can also be expanded this way:

$$A = \frac{q^2}{1-q^2} + \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \frac{4q^8}{1-q^8} + \cdots,$$

from § 42 (6.) we find:

$$(2.)\qquad 8A = \frac{(2-k^2)\left(\frac{2k}{\pi}\right)^2 - 3\frac{2K}{\pi}\cdot\frac{2E^{\mathrm{I}}}{\pi} + 1}{3}.$$

Further, it is known that:

$$8A = \frac{2}{\pi}\cdot\left(\frac{2K}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} Z\left(\frac{2Kx}{\pi}\right) Z\left(\frac{2Kx}{\pi}\right) dx;$$

for, having integrated equation (1.) from $x = 0$ to $x = \frac{\pi}{2}$, all terms except for the first vanish; hence, if one prefers to use Legendre's notation:

$$(3.)\qquad \int_0^{\frac{\pi}{2}} \frac{[F^{\mathrm{I}}E(\varphi) - E^{\mathrm{I}}F(\varphi)]^2}{\Delta(\varphi)} d\varphi = \frac{(2-k^2)F^{\mathrm{I}}F^{\mathrm{I}}F^{\mathrm{I}} - 3F^{\mathrm{I}}F^{\mathrm{I}}E^{\mathrm{I}} + \frac{1}{4}\pi\pi F^{\mathrm{I}}}{3},$$

which is the evaluation of a rather intricate definite integral.

2.4 Indefinite Elliptic Integrals of the third Kind are reduced to the definite case in which the parameter is equal to the amplitude

49.

Before we get to the expansion of the elliptic integrals of the third kind into series, we want to explain some things concerning their theory using a similar notation as Legendre. Soon the same will also be presented in new notation.

We begin with certain known theorems on elliptic integrals of the second kind. It is:

$$\sin\operatorname{am}(u+a)+\sin\operatorname{am}(u-a)=\frac{2\sin\operatorname{am}u\cos\operatorname{am}a\Delta\operatorname{am}a}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u}$$
$$\sin\operatorname{am}(u+a)-\sin\operatorname{am}(u-a)=\frac{2\sin\operatorname{am}a\cos\operatorname{am}u\Delta\operatorname{am}u}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u},$$

whence it is:

$$\sin^2\operatorname{am}(u+a)-\sin^2\operatorname{am}(u-a)=\frac{4\sin\operatorname{am}a\cos\operatorname{am}a\Delta\operatorname{am}a\sin\operatorname{am}u\cos\operatorname{am}u\Delta\operatorname{am}u}{[1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u]^2}$$

after having integrated which formula with respect to u this equation results:

$$(1.)\qquad \int\limits_0^u du[\sin^2\operatorname{am}(u+a)-\sin^2\operatorname{am}(u-a)]=\frac{2\sin\operatorname{am}a\cos\operatorname{am}a\Delta\operatorname{am}a\sin^2\operatorname{am}u}{1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u},$$

as we already found above.

Put: $\operatorname{am}u=\varphi$, $\operatorname{am}a=\alpha$, $\operatorname{am}(u+a)=\sigma$, $\operatorname{am}(u-a)=\vartheta$, in Legendre's notation it will be:

$$k^2\int\limits_0^u du\sin^2\operatorname{am}u=F(\varphi)-E(\varphi),$$

whence, because it is $F(\sigma)-F(\alpha)=F(\varphi)$, $F(\vartheta)+F(\alpha)=F(\varphi)$, it is:

$$k^2 \int_0^u du \sin^2 \operatorname{am}(u+a) = F(\varphi) - E(\sigma) + E(\alpha)$$

$$k^2 \int_0^u du \sin^2 \operatorname{am}(u-a) = F(\varphi) - E(\vartheta) - E(\alpha).$$

Hence equation (1.) goes over into this one:

$$(2.) \qquad 2E(\alpha) - [E(\sigma) - E(\vartheta)] = \frac{2k^2 \sin\alpha \cos\alpha \Delta\alpha \sin^2\varphi}{1 - k^2 \sin^2\alpha \sin^2\varphi}.$$

Having interchanged u and a, α goes over into φ, φ into $-\vartheta$, σ stays unchanged, whence from (2.) this equation results:

$$2E(\varphi) - [E(\sigma) + E(\vartheta)] = \frac{2k^2 \sin\varphi \cos\varphi \Delta\varphi \sin^2\alpha}{1 - k^2 \sin^2\alpha \sin^2\varphi},$$

having added which to equation (2.) we find:

$$(3.) \qquad E(\varphi) + E(\alpha) - E(\sigma) = k^2 \sin\alpha \sin\varphi \sin\sigma,$$

which is the theorem on the addition of the function E, given by Legendre, l.c. cap IX. pag. 43. c'.

Integrals of the form:

$$\int_0^\varphi \frac{\sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)}$$

constitute the *third* kind according to Legendre's classification of elliptic integrals into species. He calls the quantity $-k^2 \sin^2\alpha$, he denotes by n, the parameter; we will in the following call the angle α the *parameter*. For these integrals, having multiplied equation (2.) by

$$\frac{d\varphi}{\Delta(\varphi)} = \frac{d\sigma}{\Delta(\sigma)} = \frac{d\vartheta}{\Delta(\vartheta)}$$

and having integrated from $\varphi = 0$ to $\varphi = \varphi$, having done which the boundaries for σ will be: $\sigma = \alpha$, $\sigma = \sigma$, the boundaries for ϑ will be: $\vartheta = -\alpha$, $\vartheta = \vartheta$, we find the following expression:

$$\int_0^{\varphi} \frac{2k^2 \sin\sigma \cos\alpha \Delta\alpha \sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)} = 2F(\varphi)E(\alpha) - \int_{\alpha}^{\sigma} \frac{E(\sigma)d\sigma}{\Delta(\sigma)} + \int_{-\alpha}^{\varphi} \frac{E(\vartheta)d\vartheta}{\Delta(\vartheta)}.$$

It is easily seen, because it is $E(-\varphi) = -E(\varphi)$, that it will be:

$$\int_0^{\varphi} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = \int_0^{-\varphi} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} \quad \text{or} \quad \int_{-\varphi}^{\varphi} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = 0,$$

whence, because it is, it also is:

$$\int_{\alpha}^{\sigma} \frac{E(\sigma)d\sigma}{\Delta(\sigma)} = \int_0^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} - \int_0^{\alpha} \frac{E(\varphi)d\varphi}{\Delta(\varphi)}$$

$$\int_{-\alpha}^{\vartheta} \frac{E(\vartheta)d\vartheta}{\Delta(\vartheta)} = \int_0^{\vartheta} \frac{E(\sigma)d\sigma}{\Delta(\sigma)} - \int_0^{-\alpha} \frac{E(\sigma)d\sigma}{\Delta(\sigma)} = \int_0^{\vartheta} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} - \int_0^{-\alpha} \frac{E(\varphi)d\varphi}{\Delta(\varphi)},$$

we now obtain a new and memorable

Theorem I.

Determine the angles ϑ, σ that it is:

$$F(\varphi) + F(\alpha) = F(\sigma), \qquad F(\varphi) - F(\alpha) = F(\vartheta),$$

it will be:

$$\int_0^{\varphi} \frac{k^2 \sin^2\alpha \cos\alpha \Delta\alpha \sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)}$$

$$= F(\varphi)E(\alpha) - \frac{1}{2}\int_{\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = F(\varphi)E(\alpha) - \frac{1}{2}\int_0^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} + \frac{1}{2}\int_0^{\vartheta} \frac{E(\varphi)d\varphi}{\Delta(\varphi)},$$

such that the third kind of elliptic integrals depending on three variables, the modulus k, the amplitude φ, the parameter α, are reduced to the first and second kind and the new transcendent:

$$\int_0^{\varphi} \frac{E(\varphi)d\varphi}{\Delta(\varphi)},$$

which all depend only on two variables.

50.

Let us put $F(\alpha_2) = 2F(\alpha)$, if $\varphi = \alpha$, it is $\sigma = \alpha_2$, $\vartheta = 0$, in which case from the propounded theorem we obtain:

$$(1.) \qquad \int_0^{\alpha} \frac{k^2 \sin\alpha \cos\alpha \Delta\alpha \sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)} = F(\alpha)E(\alpha) - \frac{1}{2}\int_0^{\alpha_2} \frac{E(\varphi)d\varphi}{\Delta(\varphi)}.$$

This formula tells us that instead of the new transcendent one can also substitute this one:

$$\int_0^{\alpha} \frac{\sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)}$$

which is a *definite* integral of the third kind in which the amplitude is equal to the amplitude which therefore also only depends two variables, the modulus k and the quantity which is the parameter and the amplitude at the same time.

Let us put $2F(\mu) = F(\varphi) + F(\alpha) = F(\sigma)$, $2F(\delta) = F(\varphi) - F(\alpha) = F(\vartheta)$, from (1.) it will be:

$$\frac{1}{2}\int_0^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = F(\mu)E(\mu) - \int_0^{\mu} \frac{k^2 \sin\mu \cos\mu \Delta\mu \sin^2\varphi d\varphi}{[1 - k^2 \sin^2\mu \sin^2\varphi]\Delta(\varphi)}$$
$$\frac{1}{2}\int_0^{\vartheta} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = F(\delta)E(\delta) - \int_0^{\delta} \frac{k^2 \sin\delta \cos\delta \Delta\delta \sin^2\varphi d\varphi}{[1 - k^2 \sin^2\delta \sin^2\delta]\Delta(\varphi)},$$

having substituted which formulas in the theorem propounded in the preceding § we obtain the following

Theorem II.

Determine the angles μ, δ that it is:

$$F(\mu) = \frac{F(\varphi) + F(\alpha)}{2}, \qquad F(\delta) = \frac{F(\varphi) - F(\alpha)}{2},$$

it will be:

$$\begin{aligned}
k^2 \sin\alpha \cos\alpha \Delta\alpha \cdot \int_0^{\varphi} \frac{\sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)} &= F(\varphi)E(\alpha) - F(\mu)E(\mu) + F(\delta)E(\delta) \\
&+ k^2 \sin\mu \cos\mu \Delta\mu \cdot \int_0^{\mu} \frac{\sin^2\varphi d\varphi}{[1 - k^2 \sin^2\mu \sin^2\varphi]\Delta(\varphi)} \\
&- k^2 \sin\delta \cos\delta \Delta\delta \cdot \int_0^{\delta} \frac{\sin^2\varphi d\varphi}{[1 - k^2 \sin^2\delta \sin^2\varphi]\Delta(\varphi)},
\end{aligned}$$

by means of which formula the indefinite integrals of the third kind are reduced to definite ones in which the parameter becomes equal to the amplitude, and hence those indefinite integrals that depend on three variables are reduced to other transcendents that contain only two.

Having interchanged α and φ, ϑ goes over into $-\vartheta$, σ remains unchanged, whence, because moreover it is:

$$\int_{-\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)} = \int_{+\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)},$$

from theorem I:

$$\int_0^{\vartheta} \frac{k^2 \sin\alpha \cos\alpha \Delta\alpha \sin^2\varphi d\varphi}{[1 - k^2 \sin^2\alpha \sin^2\varphi]\Delta(\varphi)} = F(\varphi)E(\alpha) - \frac{1}{2}\int_{\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)}$$

we obtain:

$$\int_0^{\alpha} \frac{k^2 \sin\alpha \cos\varphi \Delta\varphi \sin^2\alpha d\alpha}{[1 - k^2 \sin^2\varphi \sin^2\alpha]\Delta(\alpha)} = F(\alpha)E(\varphi) - \frac{1}{2}\int_{\vartheta}^{\sigma} \frac{E(\varphi)d\varphi}{\Delta(\varphi)}.$$

Hence having done the calculation it results:

$$(2.)\qquad \int_0^{\vartheta} \frac{k^2 \sin\alpha \cos\alpha \Delta\alpha \sin^2\varphi d\varphi}{[1-k^2\sin^2\alpha\sin^2\varphi]\Delta(\varphi)} - \int_0^{\alpha} \frac{k^2 \sin\alpha \cos\varphi \Delta\varphi \sin^2\alpha d\alpha}{[1-k^2\sin^2\varphi\sin^2\alpha]\Delta(\alpha)} = F(\varphi)E(\alpha) - F(\alpha)E(\varphi),$$

which formula tells us that *integrals of the third kind can always be reduced to another in which what was the parameter becomes the amplitude and what was the amplitude becomes the parameter.*

If in formula (2.) one puts $\varphi = \frac{\pi}{2}$, we obtain:

$$(3.)\qquad \int_0^{\frac{\pi}{2}} \frac{k^2 \sin\alpha \cos\alpha \Delta\alpha \sin^2\varphi d\varphi}{[1-k^2\sin^2\alpha\sin^2\varphi]\Delta(\varphi)} = F^{\mathrm{I}}E(\alpha) - E^{\mathrm{I}}F(\alpha).$$

Formulas (2.), (3.) agree with those Legendre exhibited in cap. XXIII. pag. 141 (n'), (p').

2.5 The elliptic Integrals of the third kind are expanded into a series. How they are conveniently expressed by means of the new transcendent Θ

51.

From the formula:

$$\sin^2 \operatorname{am} \frac{2K}{\pi}(x+A) - \sin^2 \operatorname{am} \frac{2K}{\pi}(x-A)$$
$$= \frac{4 \sin \operatorname{am} \frac{2KA}{\pi} \cos \operatorname{am} \frac{2KA}{\pi} \Delta \operatorname{am} \frac{2KA}{\pi} \sin \operatorname{am} \frac{2Kx}{\pi} \cos \operatorname{am} \frac{2Kx}{\pi} \Delta \operatorname{am} \frac{2Kx}{\pi}}{\left\{1 - k^2 \sin^2 \operatorname{am} \frac{2KA}{\pi} \sin^2 \operatorname{am} \frac{2Kx}{\pi}\right\}^2},$$

which is known from the elements, by integrating we find:

$$(1.)\qquad \frac{2K}{\pi}\int_0^x dx \left\{\sin^2 \operatorname{am} \frac{2K}{\pi}(x+A) - \sin^2 \operatorname{am} \frac{2K}{\pi}(x-A)\right\}$$
$$= \frac{2 \sin \operatorname{am} \frac{2KA}{\pi} \cos \operatorname{am} \frac{2KA}{\pi} \Delta \operatorname{am} \frac{2KA}{\pi} \sin \operatorname{am} \frac{2Kx}{\pi}}{1 - k^2 \sin^2 \operatorname{am} \frac{2KA}{\pi} \sin^2 \operatorname{am} \frac{2Kx}{\pi}}.$$

In § 41 we already gave the formula:

$$\left(\frac{2kK}{\pi}\right)^2 \sin^2\mathrm{am}\,\frac{2Kx}{\pi} = \frac{2K}{\pi}\frac{2K}{\pi} - \frac{2K}{\pi}\frac{2E^{\mathrm{I}}}{\pi} - 4\left\{\frac{2q\cos 2x}{1-q^2} + \frac{4q^2\cos 4x}{1-q^4} + \frac{6q^3\cos 6x}{1-q^6} + \cdots\right\},$$

whence it is:

$$\begin{aligned}
&\left(\frac{2kK}{\pi}\right)^2 \left\{\sin^2\mathrm{am}\,\frac{2K}{\pi}(x+A) - \sin^2\mathrm{am}\,\frac{2K}{\pi}(x-A)\right\}\\
&= 4\left\{\frac{2q\cos 2(x-A)}{1-q^2} + \frac{4q^2\cos 4(x-A)}{1-q^4} + \frac{6q^3\cos 6(x-A)}{1-q^6} + \cdots\right\}\\
&- 4\left\{\frac{2q\cos 2(x+A)}{1-q^2} + \frac{4q^2\cos 4(x+A)}{1-q^4} + \frac{6q^3\cos 6(x+A)}{1-q^6} + \cdots\right\}\\
&= 8\left\{\frac{2q\sin 2A\sin 2x}{1-q^2} + \frac{4q^2\sin 4A\sin 4x}{1-q^4} + \frac{6q^3\sin 6A\sin 6x}{1-q^6} + \cdots\right\}.
\end{aligned}$$

Hence from (1.) it is:

$$\begin{aligned}
(2.)\qquad &\frac{2K}{\pi}\cdot\frac{2k^2\sin\mathrm{am}\,\frac{2KA}{\pi}\cos\sin\mathrm{am}\,\frac{2KA}{\pi}\Delta\sin\mathrm{am}\,\frac{2KA}{\pi}\sin^2\sin\mathrm{am}\,\frac{2Kx}{\pi}}{1-k^2\sin^2\sin\mathrm{am}\,\frac{2KA}{\pi}\sin^2\sin\mathrm{am}\,\frac{2Kx}{\pi}}\\
&= \text{const.} + 4\left\{\frac{2q\sin 2(x-A)}{1-q^2} + \frac{4q^2\sin 4(x-A)}{1-q^4} + \frac{6q^3\sin 6(x-A)}{1-q^6} + \cdots\right\}\\
&\qquad - 4\left\{\frac{2q\sin 2(x+A)}{1-q^2} + \frac{4q^2\sin 4(x+A)}{1-q^4} + \frac{6q^3\sin 6(x+A)}{1-q^6} + \cdots\right\}\\
&= \text{const.} - 8\left\{\frac{2q\sin 2A\cos 2x}{1-q^2} + \frac{4q^2\sin 4A\cos 4x}{1-q^4} + \frac{6q^3\sin 6A\cos 6x}{1-q^6} + \cdots\right\},
\end{aligned}$$

where the *constant* has to be determined in such a way that the propounded expression vanishes for $x = 0$, whence from § 47 (1.) it is:

$$\text{const.} = 8\left\{\frac{q\sin 2A}{1-q^2} + \frac{q^2\sin 4A}{1-q^4} + \frac{q^3\sin 6A}{1-q^6} + \cdots\right\} = 2\cdot\frac{2K}{\pi}Z\left(\frac{2KA}{\pi}\right).$$

Having integrated formula (2.) from $x = 0$ to $x = \frac{\pi}{2}$, because $\frac{\pi}{2}\cdot$ const. results, and the other terms vanish, having put $\frac{2KA}{\pi} = a$, $\frac{2Kx}{\pi} = u$, we find the definite integral:

$$\int\limits_0^K \frac{k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \sin^2 \operatorname{am} u du}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u} = K \cdot Z(a),$$

which is the same as (3.) § 50.

In the following we will denote the following integral by the character $\Pi(u, a, k)$ or in even shorter notation by $\Pi(u, a)$:

$$\Pi(u, a) = \int\limits_0^u \frac{k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \sin^2 \operatorname{am} u du}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u} = \int\limits_0^\varphi \frac{k^2 \sin \alpha \cos \alpha \Delta \alpha \sin^2 \varphi d\varphi}{[1 - k^2 \sin^2 \alpha \sin^2 \varphi] \Delta(\varphi)},$$

if $\varphi = \operatorname{am} u$, $\alpha = \operatorname{am} a$. Having constituted these things and integrated equation (2.) again from $x = 0$ to $x = x$ it results:

$$(3.) \qquad \Pi\left(\frac{2Kx}{\pi}, \frac{2KA}{\pi}\right)$$

$$= \frac{2Kx}{\pi} Z\left(\frac{2KA}{\pi}\right) - \left\{\frac{q \cos 2(x-A)}{1-q^2} + \frac{q^2 \cos 4(x-A)}{2(1-q^4)} + \frac{q^3 \cos 6(x-A)}{3(1-q^6)} + \cdots \right\}$$

$$+ \frac{q \cos 2(x+A)}{1-q^2} + \frac{q^2 \cos 4(x+A)}{2(1+q^4)} + \frac{q^3 \cos 6(x+A)}{3(1-q^6)} + \cdots$$

$$= \Pi\left(\frac{2Kx}{\pi}, \frac{2KA}{\pi}\right) - 2\left\{\frac{q \sin 2A \sin 2x}{1-q^2} + \frac{q^2 \sin 4A \sin 4x}{2(1-q^4)} + \frac{q^3 \sin 6A \sin 6x}{3(1-q^6)} + \cdots \right\}$$

which is the expansion of the elliptic integral of the third kind we are looking for.

If one recalls the known expansion:

$$-\ln(1 - 2q \cos 2x + q^2) = 2\left\{q \cos 2x + \frac{q^2 \cos 4x}{2} + \frac{q^3 \cos 6x}{3} + \frac{q^4 \cos 8X}{4} + \cdots \right\},$$

we see having expanded the single denominators $1 - q^2$, $1 - q^4$, $1 - q^6$ etc. that formula (3.) takes on this form:

$$(4.) \qquad \Pi\left(\frac{2Kx}{\pi}, \frac{2KA}{\pi}\right)$$

$$= \frac{2Kx}{\pi} Z\left(\frac{2KA}{\pi}\right) + \frac{1}{2} \ln\left\{\frac{(1 - 2q \cos 2(x-A) + q^2)(1 - 2q^3 \cos 2(x-A) + q^6) \cdots}{(1 - 2q \cos 2(x+A) + q^2)(1 - 2q^3 \cos 2(x+A) + q^6) \cdots}\right\}.$$

52.

Having integrated the formula (1.) § 47:

$$\frac{2K}{\pi} Z\left(\frac{2Kx}{\pi}\right) = 4\left\{\frac{q\sin 2x}{1-q^2} + \frac{q^2\sin 4x}{1-q^4} + \frac{q^3\sin 6x}{1-q^6} + \cdots\right\}$$

from $x = 0$ to $x = x$ this equation results:

$$\frac{2K}{\pi}\int_0^x Z\left(\frac{2Kx}{\pi}\right) dx = -2\left\{\frac{q\cos 2x}{1-q^2} + \frac{q^2\cos 4x}{2(1-q^4)} + \frac{q^3\sin 6x}{3(1-q^6)} + \cdots\right\} + \text{const.}$$

$$= \ln[(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots] + \text{const.}$$

where the *constant*, having determined it in such a way that the integral vanishes for $x = 0$, is:

$$2\left\{\frac{q}{1-q^2} + \frac{q^2}{2(1-q^4)} + \frac{q^3}{3(1-q^6)} + \cdots\right\} = -\ln[(1-q)(1-q^3)(1-q^5)\cdots]^2,$$

and hence:

$$(1.)\qquad \frac{2K}{\pi}\int_0^x Z\left(\frac{2Kx}{\pi}\right) dx = \ln\left\{\frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}\right\}.$$

In the following, we will denote the following expression by the character $\Theta(u)$:

$$\Theta(u) = \Theta(0)e^{\int_0^u Z(u)du},$$

where $\Theta(0)$ denotes the constant that we leave undetermined for the moment; we will find a convenient way to determine it below; from (1.) it will be:

$$(2.)\quad \frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2},$$

whence formula (4.) § 51 goes over into this one:

$$\Pi\left(\frac{2Kx}{\pi},\frac{2KA}{\pi}\right)=\frac{2Kx}{\pi}Z\left(\frac{2KA}{\pi}\right)+\frac{1}{2}\ln\frac{\Theta\left(\frac{2K}{\pi}(x-A)\right)}{\Theta\left(\frac{2K}{\pi}(x+A)\right)},$$

or having put $\frac{2Kx}{\pi}=u$, $\frac{2KA}{\pi}=a$ again:

$$(3.)\qquad \Pi(u,a)=uZ(a)+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}=u\frac{\Theta'(a)}{\Theta(a)}+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)},$$

if it is put: $\frac{d\Theta(u)}{du}=\Theta'(u)$. This is a comfortable expression for the elliptic integral Π in terms of the new transcendent Θ.

It is easily seen that $\Theta(-u)=\Theta(u)$, whence having interchanged a and u from (3.) this equation results:

$$\Pi(a,u)=aZ(u)+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)},$$

having subtracted which from (3.) it is:

$$(4.)\qquad \Pi(u,a)-\Pi(a,u)=uZ(a)-aZ(u),$$

which is the same as formula (2.) in § 50. Hence having put $\Pi(K,a)=\Pi^{\mathrm{I}}(a)$, since $\Pi(a,K)$, $Z(K)$ vanish, it is:

$$\Pi^{\mathrm{I}}(a)=KZ(a),$$

which is Legendre's formula we exhibited above as (3.) of § 50.

Having put $u=a$, from (3.) it is:

$$(5.)\qquad \Pi(a,a)=aZ(a)+\frac{1}{2}\ln\frac{\Theta(0)}{\Theta(2a)}=aZ(a)-\ln\frac{\Theta(2a)}{\Theta(0)}.$$

Therefore, we see that the new transcendent can be defined either by the integral $\int\frac{E(\varphi)d\varphi}{\Delta(\varphi)}$ by means of the formula:

$$(6.)\qquad \frac{\Theta(u)}{\Theta(0)}=e^{\int\limits_0^u Z(u)du}=e^{\int\limits_0^{\varphi}\frac{F^{\mathrm{I}}E(\varphi)-E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}\Delta(\varphi)}\cdot d\varphi},$$

or by a definite integral of the third kind by means of the formula:

$$(7.) \qquad \frac{\Theta(2a)}{\Theta(0)} = e^{2aZ(a)-2\Pi(a,a)}.$$

From formula (5.) we obtain:

$$\begin{aligned}\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)} = {} & \frac{u-a}{2}Z\left(\frac{u-a}{2}\right) - \Pi\left(\frac{u-a}{2},\frac{u-a}{2}\right) \\ & - \frac{u+a}{2}Z\left(\frac{u+a}{2}\right) + \Pi\left(\frac{u+a}{2},\frac{u+a}{2}\right),\end{aligned}$$

whence (3.) goes over into this formula:

$$\begin{aligned}(8.) \qquad \Pi(u,a) = uZ(a) + {} & \frac{u-a}{2}Z\left(\frac{u-a}{2}\right) - \frac{u+a}{2}Z\left(\frac{u+a}{2}\right) \\ & - \Pi\left(\frac{u-a}{2},\frac{u-a}{2}\right) + \Pi\left(\frac{u+a}{2},\frac{u+a}{2}\right)\end{aligned}$$

which is the formula for the reduction of an indefinite integral of the third kind to definite ones and agrees with Theorem II. § 50.

Corollary

Above we already deduced suitable algorithms for the computation from the found expansions; instead of showing new things, but to understand the nature of the things said better let us do the same for the invention of expansion of the function:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = e^{\int_0^{\varphi} \frac{F^{\mathrm{I}}E(\varphi)-E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}\Delta(\varphi)}d\varphi}$$

$$= \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}.$$

For this aim, we want to mention the following things in advance.

Put the infinite product:

$$T = \left(\frac{1-q}{1+q}\right)\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{2}}\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{4}}\left(\frac{1-q^8}{1+q^8}\right)^{\frac{1}{8}}\cdots,$$

if one substitutes the following again and again:

$$1-q^2=(1-q)(1+q), \quad 1-q^4=(1-q^2)(1+q^2), \quad 1-q^8=(1-q^4)(1+q^4), \cdots,$$

it results:

$$\begin{aligned}
T &= (1-q)\cdot\left(\frac{1-q}{1+q}\right)^{\frac{1}{2}}\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{4}}\cdot\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{8}}\cdot\left(\frac{1-q^8}{1+q^8}\right)^{\frac{1}{16}}\cdots \\
&= (1-q)\cdot\ (1-q)^{\frac{1}{2}}\ \cdot\ \left(\frac{1-q}{1+q}\right)^{\frac{1}{4}}\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{8}}\cdot\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{16}}\cdots \\
&= (1-q)\cdot\ (1-q)^{\frac{1}{2}}\ \cdot\ (1-q)^{\frac{1}{4}}\ \cdot\ \left(\frac{1-q}{1+q}\right)^{\frac{1}{8}}\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{16}}\cdots \\
&\cdots\cdots
\end{aligned}$$

whence we see that it will be:

$$(1.) \qquad T=(1-q)(1-q)^{\frac{1}{2}}(1-q)^{\frac{1}{4}}(1-q)^{\frac{1}{8}}(1-q)^{\frac{1}{16}}\cdots=(1-q)^2$$

Or also, because it is:

$$\begin{aligned}
T &= \left(\frac{1-q}{1+q}\right)\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{2}}\cdot\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{4}}\cdot\left(\frac{1-q^8}{1+q^8}\right)^{\frac{1}{8}}\cdots \\
&= \ (1-q)\ \cdot\ \left(\frac{1-q}{1+q}\right)^{\frac{1}{2}}\cdot\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{4}}\cdot\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{8}}\cdots,
\end{aligned}$$

it is $T=(1-q)\sqrt{T}$, whence $T=(1-q)^2$.

Therefore, it is

$$(2.) \qquad 1-q=\left(\frac{1-q}{1+q}\right)^{\frac{1}{2}}\left(\frac{1-q^2}{1+q^2}\right)^{\frac{1}{4}}\left(\frac{1-q^4}{1+q^4}\right)^{\frac{1}{8}}\cdots,$$

in which formula we want successively put q, q^3, q^5, q^7 etc. instead of q. Recalling and using the formula exhibited above:

$$\sqrt[4]{k'} = \left(\frac{1-q}{1+q}\right)\left(\frac{1-q^3}{1+q^3}\right)\left(\frac{1-q^5}{1+q^5}\right)\left(\frac{1-q^7}{1+q^7}\right)\cdots,$$

it results:

$$(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots = [k']^{\frac{1}{8}}[k^{(2)\prime}]^{\frac{1}{16}}[k^{(4)\prime}]^{\frac{1}{32}}\cdots,$$

if, as above, we denote the quantity which depends on q^r in the same way as k' on q or the complement of the modulus found by first transformation of r-th order by $k^{(r)\prime}$.

Further, we found in § 36:

$$\left\{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots\right\}^6 = \frac{2\sqrt[4]{q}k'}{\sqrt{k}},$$

whence now it is:

$$(3.)\qquad q = e^{\frac{-\pi K'}{K}} = \frac{kk}{16k'}[k^{(2)\prime}]^{\frac{3}{2}}[k^{(4)\prime}]^{\frac{3}{4}}[k^{(8)\prime}]^{\frac{3}{8}}\cdots$$

It is known having put $m = 1$, $n = k'$; $\frac{m+n}{2} = m'$, $\sqrt{mn} = n'$; $\frac{m'+n'}{2} = m''$, $\sqrt{m'n'} = n''$ etc. that $k^{(2)\prime} = \frac{m'}{n'}$, $k^{(4)\prime} = \frac{m''}{n''}$, $k^{(8)\prime} = \frac{m'''}{n'''}$, etc., whence:

$$(4.)\qquad q = \frac{mm-nn}{16mn}\cdot\left\{\left(\frac{n'}{m'}\right)^{\frac{1}{2}}\left(\frac{n''}{m''}\right)^{\frac{1}{4}}\left(\frac{n'''}{m'''}\right)^{\frac{1}{8}}\cdots\right\}^3.$$

Hence it also follows, because $\mu = \frac{\pi}{2K}$ denotes the common limit to which the quantities $m^{(p)}$, $n^{(p)}$ converge:

$$(5.)\qquad K' = \frac{1}{2\mu}\left\{\ln\frac{16mn}{mm-nn} + \frac{3}{2}\ln\frac{m'}{n'} + \frac{3}{4}\ln\frac{m''}{n''} + \frac{3}{8}\ln\frac{m'''}{n'''} + \cdots\right\}$$

which formulas allow a very fast calculation. (5.) tells us, how from the same series of quantities which one needs to have calculated in order to find the value of the function K the value of K' also immediately results.

Let us transform formula (3.). It is, as it is known:

$$k' = \frac{1-k^{(2)}}{1+k^{(2)}};\quad k = \frac{2\sqrt{k^{(2)}}}{1+k^{(2)}},\quad \text{whence}\quad \frac{kk}{k'} = \frac{4k^{(2)}}{k^{(2)\prime}k^{(2)\prime}}.$$

Hence, if we substitute $k^{(2)}$ for k again and again and take the square root, we obtain:

$$\begin{aligned}
\frac{kk}{16k'} \quad &\cdot \left\{k^{(2)\prime}\right\}^{\frac{3}{2}} = \left\{\frac{k^{(2)}k^{(2)}}{16k^{(2)\prime}}\right\}^{\frac{1}{2}} \\
\left\{\frac{k^{(2)}k^{(2)}}{16k^{(2)\prime}}\right\}^{\frac{1}{2}} &\cdot \left\{k^{(4)\prime}\right\}^{\frac{3}{4}} = \left\{\frac{k^{(4)}k^{(4)}}{16k^{(4)\prime}}\right\}^{\frac{1}{4}} \\
\left\{\frac{k^{(4)}k^{(4)}}{16k^{(4)\prime}}\right\}^{\frac{1}{4}} &\cdot \left\{k^{(8)\prime}\right\}^{\frac{3}{8}} = \left\{\frac{k^{(8)}k^{(8)}}{16k^{(8)\prime}}\right\}^{\frac{1}{8}} \\
\cdots,
\end{aligned}$$

whence having put $r = 2^p$ it is:

$$\frac{kk}{16k'} = \left\{k^{(2)\prime}\right\}^{\frac{3}{2}} \left\{k^{(4)\prime}\right\}^{\frac{3}{4}} \left\{k^{(8)\prime}\right\}^{\frac{3}{8}} \cdots \left\{k^{(r)\prime}\right\}^{\frac{3}{r}} = \left\{\frac{k^{(r)}k^{(r)}}{16k^{(r)\prime}}\right\}^{\frac{1}{r}}.$$

Hence we see that from formula (3.) $q = e^{\frac{-\pi K'}{K}}$ will be the limit of the expression $\left\{\frac{k^{(r)}k^{(r)}}{16k^{(r)\prime}}\right\}^{\frac{1}{r}}$ as p or r increases to infinity which is the theorem found by Legendre.

And it is immediately clear from the formula we exhibited:

$$k = 4\sqrt{q}\left\{\frac{(1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots}{(1+q)(1+q^3)(1+q^5)(1+q^7)\cdots}\right\}^4$$

that having neglected the quantities of order q^r it will be:

$$q = \sqrt[r]{\frac{k^{(r)}k^{(r)}}{16}},$$

which agrees with the mentioned theorem.

Now, in our formula:

$$1 - q = \left\{\frac{1-q}{1+q}\right\}^{\frac{1}{2}} \left\{\frac{1-q^2}{1+q^2}\right\}^{\frac{1}{4}} \left\{\frac{1-q^4}{1+q^4}\right\}^{\frac{1}{8}} \cdots$$

instead of q let us successively put the following two series of quantities:

$$\begin{array}{llll} qe^{2ix}, & q^3e^{2ix}, & q^5e^{2ix}, & q^7e^{2ix}, \cdots \\ qe^{-2ix}, & q^3e^{-2ix}, & q^5e^{-2ix}, & q^7e^{-2ix}, \cdots \end{array}$$

and multiply the infinitely many terms. Recall the formula of § 36:

$$\Delta \operatorname{am} \frac{2Kx}{\pi} = \sqrt{k'} \frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots},$$

and let us denote the following expression by $\Delta^{(r)}$

$$\Delta \operatorname{am}\left(\frac{2rK^{(r)}x}{\pi}, k^{(r)}\right) = \sqrt{k^{(r)\prime}} \frac{(1+2q^r\cos 2x+q^{2r})(1+2q^{3r}\cos 2x+q^{6r})(1+2q^{5r}\cos 2x+q^{10r})\cdots}{(1-2q^r\cos 2x+q^{2r})(1-2q^{3r}\cos 2x+q^{6r})(1-2q^{5r}\cos 2x+q^{10r})\cdots},$$

it results:

$$\frac{1}{\Delta^{(1)\frac{1}{2}}\Delta^{(2)\frac{1}{4}}\Delta^{(4)\frac{1}{8}}\Delta^{(8)\frac{1}{16}}\ldots} = \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2},$$

We determined the constant factor we added, $\frac{1}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}$, from the results found above or using that both expression for $x=0$ become equal to 1. But now we find:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2},$$

whence:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{1}{\Delta^{(1)\frac{1}{2}}\Delta^{(2)\frac{1}{4}}\Delta^{(4)\frac{1}{8}}\Delta^{(8)\frac{1}{16}}\ldots}$$

Hence having put $\frac{2kX}{\pi} = u$, $\operatorname{am} u = \varphi$ and having recalled the formulas that Legendre propounded on the transformation of the second order we obtain the following theorem which yields a fast way to the calculate of the function Θ.

Theorem

Put $\operatorname{am} u = \varphi$, $m = 1$, $n = k'$, $\Delta(\varphi) = \sqrt{mm\cos^2\varphi + nn\sin^2\varphi} = \Delta$ and calculate the series of quantities

$$\begin{array}{llll} m' = \dfrac{m+n}{2}, & m'' = \dfrac{m'+n'}{2}, & m''' = \dfrac{m''+n''}{2}, & \cdots \\ n' = \sqrt{mn}, & n'' = \sqrt{m'n'}, & n''' = \sqrt{m''n''}, & \cdots \\ \Delta' = \dfrac{\Delta\Delta + n'n'}{2\Delta}, & \Delta'' = \dfrac{\Delta'\Delta' + n''n''}{2\Delta'}, & \Delta''' = \dfrac{\Delta''\Delta'' + n'''n'''}{2\Delta''}, & \cdots \end{array}$$

it will be:

$$\frac{\Theta(u)}{\Theta(0)} = e^{\int\limits_0^{\varphi} \frac{F^{\mathrm{I}}E(\varphi) - E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}\Delta(\varphi)} d\varphi} = \left\{\frac{m}{\Delta}\right\}^{\frac{1}{2}} \cdot \left\{\frac{m'}{\Delta'}\right\}^{\frac{1}{4}} \cdot \left\{\frac{m''}{\Delta''}\right\}^{\frac{1}{8}} \cdot \left\{\frac{m'''}{\Delta'''}\right\}^{\frac{1}{16}} \cdots$$

We put aside the task to demonstrate this theorem and the consideration of expansions by means of known and finite formulas because both is easily done.

2.6 On the Addition of Arguments both of the parameter and the amplitude in the elliptic integrals of the first kind

53.

We will obtain the fundamental formula in the analysis of the function Θ, which we will use very frequently in the following, from the following consideration. For, because it was put:

$$\Pi(u,a) = \int\limits_0^u \frac{k^2 \sin\operatorname{am} a \cos\operatorname{am} a \Delta\operatorname{am} a \sin^2\operatorname{am} u du}{1 - k^2\sin^2\operatorname{am} a \sin^2 am\operatorname{am} u},$$

it is:

$$\frac{d\Pi(u,a)}{du} = \frac{k^2 \sin\operatorname{am} a \cos\operatorname{am} a \Delta\operatorname{am} a \sin^2\operatorname{am} u du}{1 - k^2\sin^2\operatorname{am} a \sin^2 am\operatorname{am} u}.$$

Having integrated this formula with respect to a from $a = 0$ to $a = a$, this equation results:

$$(1.) \qquad \int_0^a da \frac{d\Pi(u,a)}{da} = -\frac{1}{2}\ln(1 - k^2 \sin^2 \operatorname{am} \sin^2 \operatorname{am} u).$$

But from (3.) § 52 it is:

$$(2.) \qquad \frac{d\Pi(u,a)}{du} = Z(a) + \frac{1}{2}\frac{\Theta'(u-a)}{\Theta(u-a)} - \frac{1}{2}\frac{\Theta'(u+a)}{\Theta(u+a)},$$

whence:

$$\int_0^a da \frac{d\Pi(u,a)}{du} = \ln\frac{\Theta(a)}{\Theta(0)} - \frac{1}{2}\ln\Theta(u-a) - \frac{1}{2}\ln\Theta(u+a) + \ln\Theta(u),$$

having substituted which and going from logarithms to ordinary numbers from (1.) one obtains :

$$(3.) \qquad \Theta(u+a)\Theta(u-a) = \left\{\frac{\Theta(u)\Theta(a)}{\Theta(0)}\right\}^2 (1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u).$$

We can represent formula (2.) this way:

$$\frac{k^2 \sin \operatorname{am} a \cos \operatorname{am} a \Delta \operatorname{am} a \sin^2 \operatorname{am} u}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u} = Z(u) + \frac{1}{2}Z(u-a) - \frac{1}{2}Z(u+a),$$

whence having commuted a and u:

$$\frac{k^2 \sin \operatorname{am} u \cos \operatorname{am} u \Delta \operatorname{am} u \sin^2 \operatorname{am} a}{1 - k^2 \sin^2 \operatorname{am} a \sin^2 \operatorname{am} u} = Z(u) - \frac{1}{2}Z(u-a) - \frac{1}{2}Z(u+a),$$

having added which formulas this equation results:

$$(4.) \qquad Z(u) + Z(a) - Z(u+a) = k^2 \sin \operatorname{am} u \sin \operatorname{am} a \sin \operatorname{am}(u+a),$$

which is for the addition of the function Z and agrees with formula (3.) of § 49:

$$E(\varphi) + E(\alpha) - E(\sigma) = k^2 \sin\varphi \sin\alpha \sin\sigma.$$

Having put $a = K$, because it is seen that $Z(K) = \frac{F^{\mathrm{I}}E^{\mathrm{I}} - E^{\mathrm{I}}F^{\mathrm{I}}}{F^{\mathrm{I}}} = 0$, from (4.) this equation results:

$$(5.) \qquad Z(u) - Z(u+K) = k^2 \sin \mathrm{am}\, u \sin \mathrm{coam}\, u,$$

which we derived from the expansion of Z in § 47. Having put $-u$ instead of u and $K - u = v$, from formula (5.) we obtain:

$$(6.) \qquad Z(u) + Z(v) = k^2 \sin \mathrm{am}\, u \sin \mathrm{am}\, v.$$

Having put $u = v = \frac{K}{2}$ it is $2Z\left(\frac{K}{2}\right) = 1 - k'$.

Let us integrate formula (5.) from $u = 0$ to $u = u$. Because it is $\int\limits_0^u Z(u)du = \ln \frac{\Theta(u)}{\Theta(0)}$, this equation results:

$$\ln \frac{\Theta(u)}{\Theta(0)} - \ln \frac{\Theta(u+K)}{\Theta(K)} = -\ln \Delta \,\mathrm{am}\, u$$

or:

$$(7.) \qquad \frac{\Theta(0)}{\Theta(K)} \cdot \frac{\Theta(u+K)}{\Theta(u)} = \Delta \,\mathrm{am}\, u.$$

Having put $u = -K$ from (7.) we find the value of:

$$(8.) \qquad \frac{\Theta(K)}{\Theta(0)} = \frac{1}{\sqrt{k'}},$$

whence (7.) takes on the form:

$$(9.) \qquad \frac{\Theta(u+K)}{\Theta(u)} = \frac{\Delta \,\mathrm{am}\, u}{\sqrt{k'}}.$$

We easily confirm formula (9.) from the found expansion:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{(1 - 2q\cos 2x + q^2)(1 - 2q^3\cos 2x + q^6)(1 - 2q^5\cos 2x + q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}.$$

For, having changed x to $x + \frac{\pi}{2}$ it is:

$$\frac{\Theta\left(\frac{2Kx}{\pi}+K\right)}{\Theta(0)} = \frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2},$$

whence:

$$\frac{\Theta\left(\frac{2Kx}{\pi}+K\right)}{\Theta\left(\frac{2Kx}{\pi}\right)} = \frac{(1+2q\cos 2x+q^2)(1+2q^3\cos 2x+q^6)(1+2q^5\cos 2x+q^{10})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots},$$

which expression we in § 35 found to be $= \frac{\Delta\operatorname{am}\frac{2Kx}{\pi}}{\sqrt{k'}}$ as is has to be.

From formula (9.) we immediately reduce the expressions $\Pi(u+K,a)$, $\Pi(u,a+K)$ to $\Pi(u,a)$. For, it is:

$$\begin{aligned}
(10.)\qquad \Pi(u+K,a) &= (u+K)Z(a)+\frac{1}{2}\ln\frac{\Theta(u+K-a)}{\Theta(u+K+a)}\\
&= (u+K)Z(a)+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}+\frac{1}{2}\ln\frac{\Delta\operatorname{am}(u-a)}{\Delta\operatorname{am}(u+a)}\\
&= \Pi(u,a)+KZ(a)+\frac{1}{2}\ln\frac{\Delta\operatorname{am}(u-a)}{\Delta\operatorname{am}(u+a)}\\
(11.)\qquad \Pi(u,a+K) &= uZ(a+K)+\frac{1}{2}\ln\frac{\Theta(u-a-K)}{\Theta(u+a+K)}\\
&= uZ(a)-k^2\sin\operatorname{am}a\sin\operatorname{coam}a\cdot u+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}+\frac{1}{2}\ln\frac{\Delta\operatorname{am}(u-a)}{\Delta\operatorname{am}(u+a)}\\
&= \Pi(u,a)-k^2\sin\operatorname{am}a\sin\operatorname{coam}a\cdot u+\frac{1}{2}\ln\frac{\Delta\operatorname{am}(u-a)}{\Delta\operatorname{am}(u+a)}.
\end{aligned}$$

54.

From the fundamental formula, by means of which the function Π is defined by Z, Θ:

$$\text{I.}\qquad \Pi(u,a)=uZ(a)+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)},$$

and having recalled the following fundamental formulas in the analysis of the functions Z, Θ:

II. $\quad Z(u) + Z(a) - Z(u+a) = k^2 \sin \mathrm{am}\, a \sin \mathrm{am}\, u \,\mathrm{am}(u+a)$

III. $$\Theta(u+a)\Theta(u-a) \quad = \left\{ \frac{\Theta(u)\Theta(a)}{\Theta(0)} \right\}^2 (1 - k^2 \sin^2 \mathrm{am}\, a \sin^2 \mathrm{am}\, u),$$

one now easily obtains formulas both for expressing $\Pi(u+v,a)$ by means of $\Pi(u,a)$, $\Pi(v,a)$, which we will call the theorem on *the addition of the argument of the amplitude*, and for expressing $\Pi(u,a+b)$ by means of $\Pi(u,a)$, $\Pi(u,b)$, which we will call the theorem on*the addition of the argument of the parameter*. For this purpose, we add the following remarks.

From the formulas:

$$\begin{aligned}
\Pi(u,a) \quad &= uZ(a) \quad &&+ \frac{1}{2} \ln \frac{\Theta(u-a)}{\Theta(u+a)} \\
\Pi(v,a) \quad &= vZ(a) \quad &&+ \frac{1}{2} \ln \frac{\Theta(v-a)}{\Theta(v+a)} \\
\Pi(u+v,a) &= (u+v)Z(a) &&+ \frac{1}{2} \ln \frac{\Theta(u+v-a)}{.} \Theta(u+v+a)
\end{aligned}$$

it follows:

$$\Pi(u,a) + \Pi(v,a) - \Pi(u+v,a) = \frac{1}{2} \ln \frac{\Theta(u-a)\Theta(v-a)\Theta(u+v+a)}{\Theta(u+a)\Theta(v+a)\Theta(u+v-a)}. \tag{1.}$$

The expression contained in the logarithm:

$$\frac{\Theta(u-a)\Theta(v-a)\Theta(u+v+a)}{\Theta(u+a)\Theta(v+a)\Theta(u+v-a)}$$

can be reduced to elliptic functions by means of fundamental theorem (III.) in two ways. For, firstly from the theorem it is:

$$\Theta(u-a)\Theta(v-a) = \left\{\frac{\Theta\left(\frac{u-v}{2}\right)\Theta\left(\frac{u+v}{2}-a\right)}{\Theta(0)}\right\}^2 \cdot \left(1 - k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)\right)$$
$$\Theta(u+a)\Theta(v+a) = \left\{\frac{\Theta\left(\frac{u-v}{2}\right)\Theta\left(\frac{u+v}{2}+a\right)}{\Theta(0)}\right\}^2 \cdot \left(1 - k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)\right)$$
$$\Theta(u+v-a)\Theta(a) = \left\{\frac{\Theta\left(\frac{u+v}{2}\right)\Theta\left(\frac{u+v}{2}-a\right)}{\Theta(0)}\right\}^2 \cdot \left(1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)\right)$$
$$\Theta(u+v+a)\Theta(a) = \left\{\frac{\Theta\left(\frac{u+v}{2}\right)\Theta\left(\frac{u+v}{2}+a\right)}{\Theta(0)}\right\}^2 \cdot \left(1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)\right)$$

after having multiplied the first and the fourth and having divided by the second and the third of which formulas it results:

$$(2.) \qquad \frac{\Theta(u-a)\Theta(v-a)\Theta(u+v+a)}{\Theta(u+a)\Theta(v+a)\Theta(u+v-a)}$$
$$= \frac{\left\{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)\right\}\left\{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)\right\}}{\left\{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)\right\}\left\{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)\right\}}.$$

Hence, which is the other way, where fundamental theorem (III.) is represented as this:

$$\left\{\frac{\Theta(u)\theta(v)}{\Theta(0)}\right\}^2 = \frac{\Theta(u+v)\Theta(u-v)}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v},$$

it is:

$$\left\{\frac{\Theta(u-a)\Theta(v-a)}{\Theta(0)}\right\}^2 = \frac{\Theta(u-v)\Theta(u+v-2a)}{1-k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-a)}$$
$$\left\{\frac{\Theta(u+a)\Theta(v+a)}{\Theta(0)}\right\}^2 = \frac{\Theta(u-v)\Theta(u+v+2a)}{1-k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+a)}$$
$$\left\{\frac{\Theta(a)\Theta(u+v-a)}{\Theta(0)}\right\}^2 = \frac{\Theta(u+v)\Theta(u+v-2a)}{1-k^2\sin^2\operatorname{am}(a)\sin^2\operatorname{am}(u+v-a)}$$
$$\left\{\frac{\Theta(a)\Theta(u+v+a)}{\Theta(0)}\right\}^2 = \frac{\Theta(u+v)\Theta(u+v+2a)}{1-k^2\sin^2\operatorname{am}(a)\sin^2\operatorname{am}(u+v+a)},$$

after again having multiplied the first and the fourth by each other and having divided by the second and the third of which formulas and taken the square root it results:

$$(3.)\qquad \frac{\Theta(u-a)\Theta(v-a)\Theta(u+v+a)}{\Theta(u+a)\Theta(v+a)\Theta(u+v-a)}$$

$$= \sqrt{\frac{[1-k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+a)][1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}(u+v-a)]}{[1-k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-a)][1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}(u+v+a)]}}.$$

To see from the elements, how the one of the expressions (2.),(3.) can be transformed into the other, I mention the following.

If in the formula, already frequently used,

$$\sin\operatorname{am}(u+v)\sin\operatorname{am}(u-v) = \frac{\sin^2\operatorname{am}u - \sin^2\operatorname{am}v}{1-k^2\operatorname{am}u\sin^2\operatorname{am}v}$$

one puts $u+v$, $u-v$ instead of u, v, respectively, it results:

$$\sin\operatorname{am}2u\sin\operatorname{am}2v = \frac{\sin^2\operatorname{am}(u+v) - \sin^2\operatorname{am}(u-v)}{1-k^2\sin^2\operatorname{am}(u+v)\sin^2\operatorname{am}(u-v)}.$$

Further, we gave the formula:

$$\sin^2\operatorname{am}(u+v) - \sin^2\operatorname{am}(u-v) = \frac{4\sin\operatorname{am}u\cos\operatorname{am}u\Delta\operatorname{am}u\sin\operatorname{am}v\cos\operatorname{am}v\Delta\operatorname{am}v}{[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v]^2},$$

whence after the multiplication we obtain:

$$(4.)\quad 1-k^2\sin^2\operatorname{am}(u+v)\sin^2\operatorname{am}(u-v) = \frac{4\sin\operatorname{am}u\cos\operatorname{am}u\Delta\operatorname{am}u\sin\operatorname{am}v\cos\operatorname{am}v\Delta\operatorname{am}v}{\sin\operatorname{am}2u\sin\operatorname{am}2v[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v]^2}$$
$$= \frac{[1-k^2\sin^4\operatorname{am}u][1-k^2\sin^4\operatorname{am}v]}{[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v]^2}$$

by means of which formula the one of the formulas (2.), (3.) can now easily be deduced from the other.

From formula (4.) one can also deduce this more general one:

$$(5.)\qquad \frac{[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v][1-k^2\sin^2\operatorname{am}u'\sin^2\operatorname{am}v']}{[1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}u'][1-k^2\sin^2\operatorname{am}v\sin^2\operatorname{am}v']}$$

$$= \sqrt{\frac{[1-k^2\sin^2\mathrm{am}(u+u')\sin^2\mathrm{am}(u-u')][1-k^2\sin^2\mathrm{am}(v+v')\sin^2\mathrm{am}(v-v')]}{[1-k^2\sin^2\mathrm{am}(u+v)\sin^2\mathrm{am}(u-v)][1-k^2\sin^2\mathrm{am}(u'+v')\sin^2\mathrm{am}(u'-v')]}}.$$

But Legendre when he treated the addition of the argument of the amplitude (cap. XVI. *Comparison des fonctions elliptiques de la troisieme espece*) exhibited the quantity which the argument of the logarithm in this form:

$$\frac{1-k^2\sin\mathrm{am}\, a\sin\mathrm{am}\, u\sin\mathrm{am}\, v\sin\mathrm{am}(u+v-a)}{1-k^2\sin\mathrm{am}\, a\sin\mathrm{am}\, u\sin\mathrm{am}\, v\sin\mathrm{am}(u+v+a)},$$

which is not obvious at first sight how it coincides with the expressions (2.) and (3.) we found. The rather intricate transformation is done this way.

From the elementary formula we have already used very frequently it is:

$$\begin{aligned}
\sin\mathrm{am}\, u\sin\mathrm{am}\, v &= \frac{\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)-\sin^2\left(\frac{u-v}{2}\right)}{1-k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\sin^2\mathrm{am}\left(\frac{u-v}{2}\right)}\\
\sin\mathrm{am}\, a\sin\mathrm{am}(u+v-a) &= \frac{\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)-\sin^2\left(\frac{u+v}{2}-a\right)}{1-k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\sin^2\mathrm{am}\left(\frac{u+v}{2}-a\right)};
\end{aligned}$$

having multiplied them by each other, it results:

$$\begin{aligned}
&\left\{1-k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\sin^2\mathrm{am}\left(\frac{u-v}{2}\right)\right\}\left\{1-k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\sin^2\mathrm{am}\left(\frac{u+v}{2}-a\right)\right\}\\
&\qquad\qquad\times\left\{1-k^2\sin\mathrm{am}\, u\sin\mathrm{am}\, v\sin\mathrm{am}(u+v-a)\right\}\\
&=\left\{1-k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\sin^2\mathrm{am}\left(\frac{u-v}{2}\right)\right\}\left\{1-k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\sin^2\mathrm{am}\left(\frac{u+v}{2}-a\right)\right\}\\
&-k^2\left\{\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\sin^2\mathrm{am}-\left(\frac{u-v}{2}\right)\right\}\left\{\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)-\sin^2\mathrm{am}\left(\frac{u+v}{2}-a\right)\right\}
\end{aligned}$$

The other side of the equation having deleted the terms cancelling each other

$$\begin{aligned}
&-k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\left\{\sin^2\mathrm{am}\left(\frac{u-v}{2}\right)+\sin^2\mathrm{am}\left(\frac{u+v}{2}-a\right)\right\}\\
&+k^2\sin^2\mathrm{am}\left(\frac{u+v}{2}\right)\left\{\sin^2\mathrm{am}\left(\frac{u-v}{2}\right)+\sin^2\mathrm{am}\left(\frac{u+v}{2}-a\right)\right\}
\end{aligned}$$

is:

$$1+k^4\sin^4\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)$$

$$-k^2\sin^4\operatorname{am}\left(\frac{u+v}{2}\right)-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)$$

$$=\left\{1-k^2\sin^4\operatorname{am}\left(\frac{u+v}{2}\right)\right\}\left\{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)\right\},$$

whence it finally arises:

$$(6.)\qquad \frac{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)}{1-k^2\sin^4\operatorname{am}\left(\frac{u+v}{2}\right)}\left\{1-k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v-a)\right\}$$

$$=\frac{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)}{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)},$$

whence after a division:

$$(7.)\qquad \frac{1-k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v-a)}{1+k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v+a)}$$

$$=\frac{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)}{1-k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)}\cdot\frac{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}+a\right)}{1-k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2}-a\right)},$$

which is transformation in question of the expression propounded by Legendre into expression (2.).

Formula (6.) having put u, a, v instead of $\frac{u-v}{2}$, $\frac{u+v}{2}$, $\frac{u+v}{2}-a$ can also be represented this way:

$$(8.)\qquad 1-k^2\sin\operatorname{am}(a+u)\sin\operatorname{am}(a-u)\sin\operatorname{am}(a+v)\sin\operatorname{am}(a-v)$$

$$=\frac{[1-k^2\sin^4\operatorname{am}a][1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v]}{[1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}u][1-k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}v]},$$

whence formula (4.) follows as a special case having put $u=v$.

55.

From the formulas (1.), (2.), (3.), (7.) of the preceding § it follows:

$$(1.) \qquad \Pi(u,a) + \Pi(v,a) - \Pi(u+v,a)$$
$$= \frac{1}{2}\ln\frac{\left\{1 - k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right)\right\}\left\{1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} + a\right)\right\}}{\left\{1 - k^2\sin^2\operatorname{am}\left(\frac{u-v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} + a\right)\right\}\left\{1 - k^2\sin^2\operatorname{am}\left(\frac{u+v}{2}\right)\sin^2\operatorname{am}\left(\frac{u+v}{2} - a\right)\right\}}$$
$$= \frac{1}{4}\ln\frac{[1 - k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+a)][1 - k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}(u+v-a)]}{[1 - k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-a)][1 - k^2\sin^2\operatorname{am}a\sin^2\operatorname{am}(u+v+a)]}$$
$$= \frac{1}{2}\ln\frac{1 - k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v-a)}{1 + k^2\sin\operatorname{am}a\sin\operatorname{am}u\sin\operatorname{am}v\sin\operatorname{am}(u+v+a)},$$

which is the theorem on the addition of the argument of the *amplitude*. Further, by the same method one can investigate the other formula for the addition of the argument of the *parameter*, but by means of the other theorem on the reduction of the parameter to the amplitude, which formula (4.) § 52 gave us:

$$(\text{IV.}) \qquad \Pi(u,a) - \Pi(a,u) = uZ(a) - aZ(u),$$

the same follows immediately from formula (1.). For, from (IV.) it is:

$$\begin{aligned}
\Pi(a,u) &- \Pi(u,a) &&= aZ(u) &&- uZ(a)\\
\Pi(b,u) &- \Pi(u,b) &&= bZ(u) &&- uZ(b)\\
\Pi(a+b,u) &- \Pi(u,a+b) &&= (a+b)Z(u) &&- uZ(a+b),
\end{aligned}$$

whence:

$$\begin{aligned}
&\Pi(u,a) + \Pi(u,b) - \Pi(u,a+b)\\
=&\Pi(a,u) + \Pi(b,u) - \Pi(a+b,u) + u[Z(a) + Z(b) - Z(a+b)],
\end{aligned}$$

or because from (1.) it is:

$$\Pi(a,u) + \Pi(b,u) - \Pi(a+b,u) = \frac{1}{2}\ln\frac{1 - k^2\sin\operatorname{am}u\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}(a+b-u)}{1 + k^2\sin\operatorname{am}u\sin\operatorname{am}a\sin\operatorname{am}b\sin\operatorname{am}(a+b+u)},$$

further, because from (II.) it is:

$$Z(a) + Z(b) - Z(a+b) = k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b),$$

it is:

$$(2.) \qquad \Pi(u,a) + \Pi(u,b) - \Pi(u,a+b)$$
$$= k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b) \cdot u + \frac{1}{2} \ln \frac{1 - k^2 \sin\operatorname{am} u \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b-u)}{1 + k^2 \sin\operatorname{am} u \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b+u)},$$

which is the theorem in question on the addition of the argument of the *parameter*.

We will find other equally remarkable formulas by the following consideration. For, from theorem (III.) it follows:

$$\left\{ \frac{\Theta(u-a)\Theta(v-b)}{\Theta(0)} \right\}^2 = \frac{\Theta(u+v-a-b)\Theta(u-v-a+b)}{1 - k^2 \sin^2\operatorname{am}(u-a) \sin^2\operatorname{am}(v-b)}$$
$$\left\{ \frac{\Theta(u+a)\Theta(v+b)}{\Theta(0)} \right\}^2 = \frac{\Theta(u+v+a+b)\Theta(u-v+a-b)}{1 - k^2 \sin^2\operatorname{am}(u+a) \sin^2\operatorname{am}(v+b)}.$$

Now, from theorem (I.) it will be:

$$\Pi(u,a) + \Pi(v,b) = uZ/a) + vZ(b) + \frac{1}{2} \ln \frac{\Theta(u-a)\Theta(v-b)}{\Theta(u+a)\Theta(v+b)}$$
$$\Pi(u+v, a+b) + \Pi(u-v, a-b)$$
$$= (u+v)Z(a+b) + (u-v)Z(a-b) + \frac{1}{2} \ln \frac{\Theta(u+v-a-b)\Theta(u-v-a+b)}{\Theta(u+v+a+b)\Theta(u-v+a-b)},$$

whence:

$$(3.) \qquad \Pi(u+v, a+b) + \Pi(u-v, a-b) - 2\Pi(u,a) - 2\Pi(v,b)$$
$$= (u+v)Z(a+b) + (u-v)Z(a-b) - 2uZ(a) - 2vZ(b) + \frac{1}{2} \ln \frac{1 - k^2 \sin^2\operatorname{am}(u-a) \sin^2\operatorname{am}(v-b)}{1 - k^2 \sin^2\operatorname{am}(u+a) \sin^2\operatorname{am}(v+b)},$$

or because it is:

$$Z(a) + Z(b) - Z(a+b) = +k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b)$$
$$Z(a) - Z(b) - Z(a-b) = -k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a-b),$$

it will be:

$$(4.) \qquad \Pi(u+v, a+b) + \Pi(u-v, a-b) - 2\Pi(u,a) - 2\Pi(v,b)$$
$$= -k^2 \sin\operatorname{am} a \sin\operatorname{am} b [\sin\operatorname{am}(a+b)\cdot(u+v) - \sin\operatorname{am}(a-b)\cdot(u-v)]$$
$$+\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-b)}{1-k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+b)}.$$

Having interchanged u and v we obtain:

$$(5.) \qquad \Pi(u+v, a+b) - \Pi(u-v, a-b) - 2\Pi(v,a) - 2\Pi(u,b)$$
$$= -k^2 \sin\operatorname{am} a \sin\operatorname{am} b [\sin\operatorname{am}(a+b)\cdot(u+v) + \sin\operatorname{am}(a-b)\cdot(u-v)]$$
$$+\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}(v-a)\sin^2\operatorname{am}(u-b)}{1-k^2\sin^2\operatorname{am}(v+a)\sin^2\operatorname{am}(u+b)}.$$

Having added (4.) and (5.) we obtain:

$$(6.) \qquad \Pi(u+v, a+b) - \Pi(u-v, a-b) - 2\Pi(v,a) - 2\Pi(u,b)$$
$$= -k^2 \sin\operatorname{am} a \sin\operatorname{am} b \sin\operatorname{am}(a+b)\cdot(u+v)$$
$$+\frac{1}{4}\ln\left\{\frac{1-k^2\sin^2\operatorname{am}(u-a)\sin^2\operatorname{am}(v-b)}{1-k^2\sin^2\operatorname{am}(u+a)\sin^2\operatorname{am}(v+b)}\cdot\frac{1-k^2\sin^2\operatorname{am}(v-a)\sin^2\operatorname{am}(u-b)}{1-k^2\sin^2\operatorname{am}(v+a)\sin^2\operatorname{am}(u+b)}\right\}.$$

Having put $v = 0$ from (4.), (5.) it results:

$$(7.) \qquad \Pi(u, a+b) + \Pi(u, a-b) - 2\Pi(u,a)$$
$$-k^2 \sin\operatorname{am} a \sin\operatorname{am} b [\sin\operatorname{am}(a+b) - \sin\operatorname{am}(a-b)]u + \frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am} b\sin^2\operatorname{am}(u-a)}{1-k^2\sin^2\operatorname{am} b\sin^2\operatorname{am}(u+a)}$$
$$(8.) \qquad \Pi(u, a+b) - \Pi(u, a-b) - 2\Pi(u,b)$$
$$-k^2 \sin\operatorname{am} a \sin\operatorname{am} b [\sin\operatorname{am}(a+b) - \sin\operatorname{am}(a-b)]u + \frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am} a\sin^2\operatorname{am}(u-b)}{1-k^2\sin^2\operatorname{am} a\sin^2\operatorname{am}(u+b)}.$$

Having put $b = 0$, from (4.), (5.) these equations result:

$$(9.)\qquad \Pi(u+v,a)+\Pi(u-v,a)-2\Pi(u,a)=\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}v\sin^2\operatorname{am}(u-a)}{1-k^2\sin^2\operatorname{am}v\sin^2\operatorname{am}(u+a)}$$

$$(10.)\qquad \Pi(u+v,a)-\Pi(u-v,a)-2\Pi(v,a)=\frac{1}{2}\ln\frac{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}(v-a)}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}(v+a)}.$$

2.7 Reductions of the Expressions $Z(iu)$, $\Theta(iu)$ to a real Argument. The general reduction of elliptic Integrals of the third Kind, in which the Arguments both of the Amplitude and the Parameter are imaginary

56.

We return to the analysis of the functions Z, Θ whose extraordinary use we demonstrated in the preceding paragraphs. Let the reduction of the expressions $Z(iu)$, $\Theta(iu)$ to a real argument be in question. We pursue the same at first using Legendre's notation, then we will use our notation.

We know from the elements, § 19, that the following equations hold:

$$\sin\varphi == i\tan\Psi,\qquad \frac{d\varphi}{\Delta(\varphi)}=\frac{id\psi}{\Theta(\psi,k')},\qquad F(\varphi)=iF(\psi,k').$$

Hence it is:

$$d\Delta(\varphi)=\frac{id\psi(1+kk\tan^2\psi)}{\Delta(\psi,k')}=\frac{id\psi\Delta(\psi,k')}{\cos^2\psi},$$

whence after an integration:

$$\int\limits_0^\varphi\Delta(\varphi)d\varphi=i\left\{\tan\psi\Delta(\psi,k')+\int\limits_0^\psi\frac{k'k'\sin^2\psi}{\Delta(\psi,k')}d\psi\right\}$$

or:

$$(1.)\qquad E(\varphi)=i[\tan\psi\Delta(\psi,k')+F(\psi,k')-E(\psi,k')].$$

By multiplying by $\frac{d\varphi}{\Delta(\varphi)}=\frac{id\psi}{\Delta(\psi,k')}$ and by integration we find:

$$(2.)\qquad \int\limits_0^\varphi\frac{E(\varphi)}{\Delta(\varphi)}d\varphi=\ln\cos\psi-\frac{1}{2}\left\{F(\psi,k')\right\}^2+\int\limits_0^\psi\frac{E(\psi,k')}{\Delta(\psi,k')}d\psi.$$

From equation (1.) it follows:

$$\frac{F^{\mathrm{I}}E(\varphi)E^{\mathrm{I}}F(\varphi)}{i} = F^{\mathrm{I}}\tan(\psi)\Delta(\psi,k') - [F^{\mathrm{I}}E(\psi,k') - (E^{\mathrm{I}} - F^{\mathrm{I}})F(\psi,k')].$$

Now, note the extraordinary theorem due to Legendre (pag. 61):

$$F^{\mathrm{I}}E^{\mathrm{I}}(k') + F^{\mathrm{I}}(k')E^{\mathrm{I}} - F^{\mathrm{I}}F^{\mathrm{I}}(k') = \frac{\pi}{2},$$

whence it follows:

$$F^{\mathrm{I}}E(\psi,k') + (E^{\mathrm{I}} - F^{\mathrm{I}})F(\psi,k') = \frac{F^{\mathrm{I}}}{F^{\mathrm{I}}(k')}[F^{\mathrm{I}}(k')E(\psi,k') - E^{\mathrm{I}}(k')F(\psi,k')] + \frac{\pi F(\psi,k')}{2F^{\mathrm{I}}(k')},$$

and hence:

$$(3.) \qquad \frac{F^{\mathrm{I}}E(\varphi) - E^{\mathrm{I}}F(\varphi)}{iF^{\mathrm{I}}} = \tan\psi\Delta(\psi,k') - \frac{F^{\mathrm{I}}(k')E(\psi,k') - E^{\mathrm{I}}(k')F(\psi,k')}{F^{\mathrm{I}}(k')} - \frac{\pi F(\psi,k')}{2F^{\mathrm{I}}F^{\mathrm{I}}(k')}.$$

In our notation it was:

$$\varphi = \mathrm{am}(iu), \quad \psi = \mathrm{am}(u,k'), \quad E(\varphi) = iu, \quad F(\psi,k') = u;$$

further,

$$\frac{F^{\mathrm{I}}E(\varphi) - E^{\mathrm{I}}F(\varphi)}{F^{\mathrm{I}}} = Z(iu,k), \quad \frac{F^{\mathrm{I}}(k')E(\psi,k') - E^{\mathrm{I}}(k')F(\psi,k')}{F^{\mathrm{I}}(k')} = Z(u,k'),$$

whence equation (3.) is also represented this way:

$$(4.) \qquad iZ(iu,k) = -\tan\mathrm{am}(u,k')\Delta(u,k') + \frac{\pi u}{2KK'} + Z(u,k').$$

Hence by integration this equation follows:

$$\int_0^u iduZ(iu,k) = \ln\cos\mathrm{am}(u,k') + \frac{\pi uu}{4KK'} + \int_0^u Z(u,k')du,$$

or, because it is $\int_0^u duZ(u) = \ln\frac{\Theta(u)}{\Theta(0)}$:

$$(5.) \qquad \frac{\Theta(iu,k)}{\Theta(0,k)} = e^{\frac{\pi uu}{4KK'}}\cos\mathrm{am}(u,k')\frac{\Theta(u,k')}{\Theta(0,k')}.$$

Formulas (4.), (5.) reduce the functions $Z(iu)$, $\Theta(iu)$ to a real argument.

57.

In (5.) of the preceding § change u into $u + 2K'$, it results:

$$\frac{\Theta(iu + 2iK')}{\Theta(0)} = -e^{\frac{\pi(u+2K')^2}{4KK'}} \cos\mathrm{am}(u, k') \frac{\Theta(u, k')}{\Theta(0, k)} = -e^{\frac{\pi(K'+u)}{K}} \frac{\Theta(iu)}{\Theta(0)},$$

or having put u instead of iu:

$$(1.) \qquad \Theta(u + 2iK) = -e^{\frac{\pi(K'-iu)}{K}} \Theta(u).$$

In (5.) of the preceding § put $u + K'$ instead of u: Because it is:

$$\begin{aligned} \cos\mathrm{am}(u + K', k') &= -\frac{k \sin\mathrm{am}(u, k'}{\Delta\,\mathrm{am}(u, k'} \\ \Theta(u + K', k') &= \frac{\Delta\,\mathrm{am}(u, k')}{\sqrt{k}} \Theta(u, k'), \end{aligned}$$

confer § 53 (9.), this gives us the equations:

$$\begin{aligned} \frac{\Theta(iu + iK')}{\Theta(0)} &= -e^{\frac{\pi(u+K')^2}{4KK'}} \sqrt{k} \sin\mathrm{am}(u, k') \frac{\Theta(u, k')}{\Theta(0, k')} \\ &= -e^{\frac{\pi(2u+K')^2}{4K}} \sqrt{k} \tan\mathrm{am}(u, k') \frac{\Theta(iu)}{\Theta(0)}, \end{aligned}$$

whence having put u instead of iu again it is:

$$(2.) \qquad \Theta(u + iK') = ie^{\frac{\pi(K'-2iu)}{4K}} \sqrt{k} \sin\mathrm{am}\, u \Theta(u).$$

Having taken logarithms and by differentiating from (1.) and (2.) these equations result:

$$\begin{aligned} &(3.) \qquad Z(u + 2iK') = \frac{-i\pi}{K} + Z(u) \\ &(4.) \qquad Z(u + iK') = \frac{-i\pi}{2K} + \cot\mathrm{am}\, u \Delta\,\mathrm{am}\, u + Z(u). \end{aligned}$$

Having put $u = 0$ from (1.) – (4.) it follows:

$$
(5.) \qquad \begin{cases} \Theta(2iK') = -e^{\frac{\pi K'}{K}}\Theta(0), & \Theta(iK') = 0 \\ Z(2iK') = \dfrac{-i\pi}{K}, & Z(iK') = \infty \end{cases}
$$

Formulas (1.), (2.) are confirmed using the infinite products into which we expanded the function Θ:

$$
(6.) \qquad \frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}
$$

$$
= \frac{[(1-qe^{2ix})(1-q^3e^{2ix})(1-q^5e^{2ix})\cdots][(1-qe^{-2ix})(1-q^3e^{-2ix})(1-q^5e^{-2ix})\cdots]}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}
$$

For, if x is changed into $x+\frac{i\pi K'}{K}$ having done which e^{ix} goes over into qe^{ix} the product:

$$
[(1-qe^{2ix})(1-q^3e^{2ix})(1-q^5e^{2ix})\cdots][(1-qe^{-2ix})(1-q^3e^{-2ix})(1-q^5e^{-2ix})\cdots]
$$

goes over into this one:

$$
\frac{-1}{qe^{2ix}}[(1-qe^{2ix})(1-q^3e^{2ix})(1-q^5e^{2ix})\cdots][(1-qe^{-2ix})(1-q^3e^{-2ix})(1-q^5e^{-2ix})\cdots],
$$

whence:

$$
(7.) \qquad \Theta\left(\frac{2Kx}{\pi}+2iK'\right) = -\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{q^{2ix}}.
$$

On the other hand, having changed x into $x+\frac{i\pi K'}{2K}$, e^{ix} goes over into $\sqrt{q}e^{ix}$, whence the product:

$$
[(1-qe^{2ix})(1-q^3e^{2ix})(1-q^5e^{2ix})\cdots][(1-qe^{-2ix})(1-q^3e^{-2ix})(1-q^5e^{-2ix})\cdots]
$$

goes over into this one:

$$
(1-e^{-2ix})[(1-q^2e^{2ix})(1-q^4e^{2ix})\cdots][(1-q^2e^{-2ix})(1-q^4e^{-2ix})\cdots]
$$

$$= \frac{i}{e^{ix}} \cdot 2\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots$$

But in § 36 we gave the formula:

$$\sin\operatorname{am}\frac{2Kx}{\pi} = \frac{1}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots},$$

whence we see that it will be:

$$(8.)\qquad \Theta\left(\frac{2Kx}{\pi}+iK'\right) = \frac{i\sqrt{k}\sin\operatorname{am}\frac{2Kx}{\pi}\Theta\left(\frac{2Kx}{\pi}\right)}{\sqrt[4]{q}e^{ix}}.$$

But, formulas (7.), (8.) having put $\frac{2Kx}{\pi} = u$ agree with formulas (1.), (2.)

From formula (9.) § 53:

$$\Theta(u+K) = \frac{\Delta\operatorname{am}u}{\sqrt{k'}}\cdot\Theta(u),$$

having put iu instead of u, it follows:

$$\Theta(iu+K) = \frac{\Delta\operatorname{am}(u,k')}{\sqrt{k'}\cos\operatorname{am}(u,k')}\cdot\Theta(iu),$$

whence from (5.) of § 56:

$$\frac{\Theta(iu+K)}{\Theta(0)} = \frac{1}{\sqrt{k'}}e^{\frac{\pi uu}{4KK'}}\Delta\operatorname{am}(u,k')\cdot\frac{\Theta(u,k')}{\Theta(0,k')}$$

or from the aforementioned formula (9.) § 53:

$$(9.)\qquad \frac{\Theta(iu+K)}{\Theta(0)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi uu}{4KK'}}\frac{\Theta(u+K',k')}{\Theta(0,k')}.$$

Hence by taking logarithms and differentiating them we obtain:

$$(10.)\qquad iZ(iu+K) = \frac{\pi u}{2KK'} + Z(u+K',k').$$

58.

The formulas found in §§ 56 and 57 have a simple application to the analysis of the functions Π in the cases in which the arguments either of the amplitude or of the parameter or even of both are imaginary.

At first, let us demonstrate that the expression $\Pi(u, a+iK')$ can be reduced to $\Pi(u, a)$ whence it is clear that having put $n = -k^2 \sin^2 \operatorname{am} a$ the integrals:

$$\int_0^\varphi \frac{d\varphi}{(1+n\sin^2\varphi)\Delta(\varphi)}, \quad \int_0^\varphi \frac{d\varphi}{\left(1+\frac{k^2}{n}\sin^2\varphi\right)\Delta(\varphi)}$$

depend on each other; this is an extraordinary theorem stated by Legendre in cap. XV.

We found:

$$\Pi(u, a+iK') = uZ(a+iK') + \frac{1}{2}\ln\frac{\Theta(a-u+iK')}{\Theta(a+u+iK')}.$$

But from (2.), (4.) of § 57 it is:

$$\frac{\Theta(a-u+iK')}{\Theta(a+u+iK')} = e^{\frac{i\pi u}{K}} \frac{\sin\operatorname{am}(a-u)}{\sin\operatorname{am}(a-u)} \cdot \frac{\Theta(a-u)}{\Theta(a+u)}$$

$$uZ(a+iK') = -\frac{i\pi u}{2K} + u\cot\operatorname{am} a\Delta\operatorname{am} a + uZ(a),$$

whence, since the terms $\frac{i\pi u}{2K}, -\frac{i\pi u}{2K}$ cancel, it is:

$$(1.) \qquad \Pi(u, a+iK') = \Pi(u, a) + u\cot\operatorname{am} a\Delta\operatorname{am} a + \frac{1}{2}\ln\frac{\sin\operatorname{am}(a-u)}{\sin\operatorname{am}(a+u)}.$$

Let us put ia instead of a in this formula, it is:

$$\cot\operatorname{am} ia\Delta\operatorname{am} ia = \frac{-i\Delta\operatorname{am}(a.k')}{\sin\operatorname{am}(a,k')\cos\operatorname{am}(a,k')}$$

$$\frac{\sin\operatorname{am}(ia-u)}{\sin\operatorname{am}(ia+u)} = \frac{\Delta\operatorname{am} u - \cot\operatorname{am} ia\Delta\operatorname{am} ia\tan\operatorname{am} u}{\Delta\operatorname{am} u + \cot\operatorname{am} ia\Delta\operatorname{am} ia\tan\operatorname{am} u},$$

or having for the sake of brevity put :

$$\frac{\Delta\operatorname{am}(a,k')}{\sin\operatorname{am}(a,k')\cos\operatorname{am}(a,k')} = \sqrt{\alpha},$$

it is:

$$\frac{\sin\operatorname{am}(ia-u)}{\sin\operatorname{am}(ia+u)} = \frac{\Delta\operatorname{am}u + i\sqrt{\alpha}\tan\operatorname{am}u}{\Delta\operatorname{am}u - i\sqrt{\alpha}\tan\operatorname{am}u},$$

whence (1.) goes over into:

$$(2.)\qquad \frac{\Pi(u, ia+iK') - \Pi(u, ia)}{i} = -\sqrt{\alpha}\cdot u + \arctan\frac{\sqrt{\alpha}\tan\operatorname{am}u}{\Delta\operatorname{am}u},$$

which agrees with formula (f') exhibited by Legendre.

59.

We obtain other formulas, fundamental for the reduction of an imaginary argument to a real one, from (9.), (10.). First, I mention that one of those by means of which imaginary arguments of both the amplitude and the parameter are reduced to real arguments

$$(1.)\qquad \Pi(iu, ia+K) = \Pi(u, a+K', k'),$$

which is demonstrated this way. For, it is:

$$\Pi(iu, ia+K) = iuZ(ia+K) + \frac{1}{2}\ln\frac{\Theta(ia-iu+K)}{\Theta(ia+iu+K)};$$

further, from (10.) § 57 it:

$$iuZ(ia+K) = \frac{\pi ua}{2KK'} + uZ(a+K', k'),$$

from (9.) § 57 it is:

$$\frac{\Theta(ia-iu+K)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi(a-u)^2}{4KK'}}\frac{\Theta(a-u+K', k')}{\Theta(0,k')}$$
$$\frac{\Theta(ia+iu+K)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi(a+u)^2}{4KK'}}\frac{\Theta(a+u+K', k')}{\Theta(0,k')},$$

whence it is:

$$\frac{\Theta(ia-iu+K)}{\Theta(ia+iu+K)} = e^{\frac{-\pi au}{KK'}}\frac{\Theta(a-u+K', k')}{\Theta(a+u+K', k')}$$

and hence cancelling the terms $\frac{\pi ua}{2KK'}$, $-\frac{\pi ua}{2KK'}$ it is:

$$\Pi(iu, ia+K) = uZ(a+K', k') + \frac{1}{2}\ln\frac{\Theta(a-u+K', k')}{\Theta(a+u+K', k')} = \Pi(u, a+K', k'),$$

which was to be demonstrated.

Having changed a to $-ia$ in (1.) it results:

$$(2.) \qquad \Pi(iu, a+K) = -\Pi(u, ia+K', k').$$

Formula (1.) is also easily proved considering the integral by means of which we defined the function Π:

$$\Pi(u, a) = \int\limits_0^u \frac{k^2 \sin\operatorname{am} a \cos\operatorname{am} a \Delta\operatorname{am} a \sin^2\operatorname{am} u}{1 - k^2\sin^2\operatorname{am} a \sin^2\operatorname{am} u} du,$$

whence it is:

$$\Pi(iu, ia+K) = \int\limits_0^u \frac{ik^2 \sin\operatorname{am}(ia+K)\cos\operatorname{am}(ia+K)\Delta\operatorname{am}(ia+K)\sin^2\operatorname{am} iu}{1 - k^2\sin^2\operatorname{am}(ia+K)\sin^2\operatorname{am} u} du.$$

For, from the formulas of § 19 it is:

$$\begin{aligned}
\sin\operatorname{am}(ia+K) &= +\sin\operatorname{coam} ia = \frac{\Delta\operatorname{coam}(a, k')}{k} &&= \frac{\Delta\operatorname{am}(a+K', k')}{k} \\
\cos\operatorname{am}(ia+K) &= -\cos\operatorname{coam} ia = \frac{-ik'}{k}\cos\operatorname{coam}(a, k') &&= \frac{ik'}{k}\cos\operatorname{am}(a+K', k') \\
\Delta\operatorname{am}(ia+K) &= \Delta\operatorname{coam} ia = k'\sin\operatorname{coam}(a, k') &&= k'\sin\operatorname{am}(a+K', k'),
\end{aligned}$$

whence it is:

$$\begin{aligned}
&+ikk\sin\operatorname{am}(ia+K)\cdot\cos\operatorname{am}(ia+K)\cdot\Delta\operatorname{am}(ia+K) \\
&= -k'k'\sin\operatorname{am}(a+K', k')\cos\operatorname{am}(a+K', k')\Delta\operatorname{am}(a+K', k').
\end{aligned}$$

Further, it is:

$$\frac{\sin^2 \operatorname{am} iu}{1 - k^2 \sin^2 \operatorname{am}(ia + K) \sin^2 \operatorname{am} iu} = \frac{-tan^2 \operatorname{am}(u, k')}{1 + \Delta \operatorname{am}(a + K', k') \tan^2 \operatorname{am}(u, k')}$$

$$= \frac{-\sin^2 \operatorname{am}(u, k')}{\cos^2 \operatorname{am}(u, k') + \Delta^2 \operatorname{am}(a + K', k') \sin^2 \operatorname{am}(u, k')} = \frac{-\sin^2 \operatorname{am}(u, k')}{1 - k'k' \sin^2 \operatorname{am}(a + K', k') \sin^2 \operatorname{am}(u, k')},$$

whence it is:

$$\Pi(iu, ia + K) = \int_0^u \frac{k'k' \sin \operatorname{am}(a + K', k') \cos \operatorname{am}(a + K', k') \Delta \operatorname{am}(a + K', k') \sin^2 \operatorname{am}(u, k')}{1 - k'k' \sin^2 \operatorname{am}(a + K', k') \sin^2 \operatorname{am}(u, k')} du,$$

or:

$$\Pi(iu, ia + K) = \Pi(u, a + K', k'),$$

what was to be proved.

From formulas (9.), (10.) of § 57 in the same way as (1.) we can prove the following formula which tells us that two functions of an imaginary argument of the parameter of which the one is the complement of the other modulus can be reduced to each other:

$$(3.) \qquad i\Pi(u, ia + K) + i\Pi(a, iu + K', k') = \frac{\pi au}{2KK'} + uZ(a + K', k') + aZ(u + K, k).$$

For, it is:

$$\begin{aligned} i\Pi(u, ia + K) &= iuZ(ia + K) + \frac{i}{2} \ln \frac{\Theta(ia + K - u)}{\Theta(ia + K + u)} \\ i\Pi(a, iu + K', k') &= iaZ(iu + K', k') + \frac{i}{2} \ln \frac{\Theta(iu + K' - a, k')}{\Theta(iu + K' + a, k')}. \end{aligned}$$

Now it is:

$$\begin{aligned} \frac{\Theta(ia + K - u)}{\Theta(0)} &= \frac{\Theta(i(a + iu) + K)}{\Theta(0)} = \sqrt{\frac{k}{k'}} e^{\frac{\pi(a+iu)^2}{4KK'}} \frac{\Theta(a + iu + K', k')}{\Theta(0, k')} \\ \frac{\Theta(ia + K + u)}{\Theta(0)} &= \frac{\Theta(i(a - iu) + K)}{\Theta(0)} = \sqrt{\frac{k}{k'}} e^{\frac{\pi(a-iu)^2}{4KK'}} \frac{\Theta(a - iu + K', k')}{\Theta(0, k')} \end{aligned}$$

whence because it is $\Theta(u+K) = \Theta(K-u)$ it also is:

$$\frac{\Theta(ia+K-u)}{\Theta(ia+K+u)} = e^{\frac{i\pi au}{KK'}} \frac{\Theta(iu+K'+a,k')}{\Theta(iu+K'-a,k')},$$

and hence:

$$\frac{i}{2}\ln\frac{\Theta(ia+K-u)}{\Theta(ia+K+u)} + \frac{i}{2}\ln\frac{\Theta(iu+K'-a,k')}{\Theta(iu+K'+a,k')} = -\frac{\pi au}{2KK'}.$$

Further, it is:

$$\begin{aligned} iuZ(ia+K) \quad &= \frac{\pi au}{2KK'} + uZ(a+K',k') \\ iaZ(iu+K',k') &= \frac{\pi au}{2KK'} + aZ(u+K,k), \end{aligned}$$

whence:

$$i\Pi(u,ia+K) + i\Pi(a,iu+K',k') = \frac{\pi au}{2KK'} + uZ(a+K',k') + aZ(u+K,k),$$

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60.

It is clear from the formulas:

$$\begin{aligned} \sin\operatorname{am}(K+iu) \; &= \frac{1}{k}\Delta\operatorname{coam}(u,k') \\ \sin\operatorname{am}(u+iK') &= \frac{1}{k}\cdot\frac{1}{\sin\operatorname{am} u}, \end{aligned}$$

that the argument u which as $\sin\operatorname{am} u$ increases from 0 to 1 increases from 0 to K, if $\sin\operatorname{am} u$ goes over from 1 to $\frac{1}{k}$, takes on an imaginary value of the form $K+iv$ such that at the same time v increases from 0 to K'; after this, while $\sin\operatorname{am} u$ increases from $\frac{1}{k}$ to ∞, u takes on the form $v+iK'$ such that at the same time v decreases from K to 0.

Hence we see that, if in the elliptic integrals of the third kind, which is contained in this scheme:

$$\int\limits_0^{\varphi} \frac{d\varphi}{(1+n\sin^2\varphi)\Delta(\varphi)},$$

one puts, as we did, $n = -\sin^2 \sin^2 \operatorname{am} a$, if n is negative number

between	$+0$	and	$-kk$,	one has to put	$n = -k^2 \sin^2 \operatorname{am} a$
between	$-kk$	and	-1,	one has to put	$n = -k^2 \sin^2 \operatorname{am}(ia + K)$
between	-1	and	$-\infty$,	one has to put	$n = -k^2 \sin^2 \operatorname{am}(a + iK')$,

while a denotes a real quantity. Further, because it is $-kk \sin^2 \operatorname{am} ia = kk \tan \tan^2 \operatorname{am}(a, k')$, it is clear, if n is an arbitrary positive number, that one has to put:

$$n = -kk \sin^2 \operatorname{am} ia.$$

Hence we obtained four classes of elliptic integrals of the third kind corresponding to the schemes which take on the following arguments:

$$(1) \quad a, \qquad 2) \quad ia + K, \qquad 3) \quad a + iK', \qquad 4) \quad ia,$$

of which the first three correspond to a negative n, the fourth to a positive n.

But, from formula (1.) of § 58 we see that the function $\Pi(u, a + iK')$ can be reduced to $\Pi(u, a)$, or the third class, in which n lies between -1 and $-\infty$, can be reduced to the first, in which n lies between 0 and $-kk$. Further, from formula (11.) of § 53 we see that the function $\Pi(u, ia)$ can always be reduced to $\Pi(u, ia + K)$ or the fourth class in which n is positive can be reduced to the second in which n is negative between $-kk$ and -1. Hence we now obtained the theorem that *the propounded integral:*

$$\int\limits_0^\varphi \frac{d\varphi}{(1 + n \sin^2 \varphi)\Delta(\varphi)},$$

whatever real positive or negative number n is, can always be reduced to a similar integral in which n is negative and lies between 0 *and* 1. This is the extraordinary discovery of Legendre.

But now let us consider the general case in which both the amplitude and the parameter have an arbitrary imaginary form: It is clear that this case contains the expression:

$$\Pi(u+iv, a+ib),$$

u, v, a, b denoting real numbers. But from the formulas of § 55 we see that an expression of such a kind can be reduced to these four:

$$1)\quad \Pi(u,a), \qquad 2)\quad \Pi(iv,ib), \qquad 3)\quad \Pi(u,ib), \qquad 4)\quad \Pi(iv,a),$$

or, if it pleases, to these four:

$$1)\quad \Pi(u,a-K), \qquad 2)\quad \Pi(iv,ib+K), \qquad 3)\quad \Pi(u,ib+K), \qquad 4)\quad \Pi(iv,a-K).$$

For, in general the expression $\Pi(u+v, a+b)$ reduces to $\Pi(u,a)$, $\Pi(v,b)$, $\Pi(u,b)$, $\Pi(v,a)$ from which the four propounded formulas result, if one puts iv instead of v, but $a-K$, $K+ib$ instead of a,b, respectively. Further, from the formulas (1.), (2.) of § 59 it follows:

$$\begin{aligned}\Pi(iv, ib+K) &= +\Pi(v, b+K', k') \\ \Pi(iv, a-K) &= -\Pi(v, ia+K', k'),\end{aligned}$$

whence expressions 1), 2) reduce to the first class $\Pi(u,a)$, the expressions 3), 4) reduce to the second class $\Pi(u, ia+K)$; this gives us the following

Theorem

The propounded integral of the form

$$\int\limits_0^{\varphi} \frac{d\varphi}{(1+n\sin^2\varphi)\Delta(\varphi)},$$

whatever n and φ are, either real or imaginary, can be reduced to similar integrals in which φ is real and n is real and lies between 0 and -1.

This theorem is also due to Legendre except that he considered only real amplitudes.

By means of formulas (4.), (5.) of § 55 $\Pi(u+v, a+b) + \Pi(u-v, a-b)$ is reduced to $\Pi(u,a)$ and $\Pi(v,b)$, $\Pi(u+v,a+b)-\Pi(u-v,a-b)$ is reduced to $\Pi(u,b)$ and $\Pi(v,a)$. Hence it is clear that having put:

$$\begin{aligned}\Pi(u+iv,a+ib)+\Pi(u-iv,a-ib) &= L,\\ \frac{\Pi(u+iv,a+ib)-\Pi(u-iv,a-ib)}{i} &= M,\end{aligned}$$

L depends on the functions $\Pi(u,a-K)$, $\Pi(iv,ib+K)$, M depends on the functions $\Pi(u,ib+K)$, $\Pi(iv,a-K)$ and hence L reduces to the first class, M reduces to the second class.

These are the foundations of the theory of the elliptic integrals of the third kind, deduced from new principles. We will see others below.

2.8 Elliptic Functions are rational Functions. On the Functions H, Θ which constitute the numerator and the denominator, respectively.

61.

The expansions exhibited in § 35 reveal the genuine nature of elliptic functions, of course that they are rational functions, and, as we already know from the elements, that they vanish and become infinite for innumerable different values of the argument. We have already been led to the function constituting the denominator of the fraction, into which we expanded it

$$\sin\operatorname{am}\frac{2kx}{\pi}=\frac{1}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots},$$

in the preceding paragraphs, I mean the function:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)}=\frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots]^2}.$$

Now, let us also denote the numerator by a particular character, and let us put:

$$\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta(0)}=\frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{[(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots]^2}.$$

it will be:

$$\sin\operatorname{am}\frac{2Kx}{\pi} = \frac{1}{\sqrt{k}}\cdot\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}.$$

Recalling the expansions given in § 36, we find:

$$\begin{aligned}
\cos\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{\frac{k'}{k}}\cdot\frac{H\left(\frac{2K}{\pi}\left(x+\frac{\pi}{2}\right)\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}\\
\Delta\operatorname{am}\frac{2Kx}{\pi} &= k'\cdot\frac{\Theta\left(\frac{2K}{\pi}\left(x+\frac{\pi}{2}\right)\right)}{\Theta\left(\frac{2Kx}{\pi}\right)},
\end{aligned}$$

whence having put $\frac{2Kx}{\pi} = u$ it is:

$$(1.)\quad \sin\operatorname{am}u = \frac{1}{\sqrt{k}}\cdot\frac{H(u)}{\Theta(u)};\quad \cos\operatorname{am}u = \sqrt{\frac{k'}{k}}\cdot\frac{H(u+K)}{\Theta(u)};\quad \Delta\operatorname{am}u = \sqrt{k'}\cdot\frac{\Theta(u+K)}{\Theta(u)}.$$

Hence these special formulas follow:

$$(2.)\qquad \Theta(K) = \frac{\Theta(0)}{\sqrt{k'}},\quad H(K) = \sqrt{\frac{k}{k'}}\Theta(0).$$

Having put $H'(u) = \frac{dH(u)}{du}$, because it is:

$$H'(u) = \sqrt{k}\cos\operatorname{am}u\;\Delta\operatorname{am}u\Theta(u) + \sqrt{k}\sin\operatorname{am}u\;\Theta'(u),$$

for the values $u = 0$, $u = K$ we obtain:

$$(3.)\quad H'(0) = \sqrt{k}\Theta(0) = \frac{H(K)\Theta(0)}{\Theta(K)};\quad H'(K) = \sqrt{k}\Theta'(K) = 0.$$

From (2.) it also follows:

$$(4.)\qquad \sqrt{k} = \frac{H(K)}{\Theta(K)};\qquad \sqrt{k'} = \frac{\Theta(0)}{\Theta(K)}.$$

Moreover, it is:

$$\begin{aligned}
&(5.)\qquad \Theta(u+2K) = \Theta(-u) = \Theta(u)\\
&(6.)\qquad H(u+2K) = H(-u) = -H(u);\quad H(u+4K) = H(u);
\end{aligned}$$

From formula (2.) of § 57:

$$\Theta(u+iK') = ie^{\frac{\pi(K'-2iu)}{4K}}\sqrt{k}\sin\operatorname{am} u\ \Theta(u)$$

it follows:

$$(7.)\qquad \Theta(u+iK') = ie^{\frac{\pi(K'-2iu)}{4K}}H(u).$$

Having changed u into $u+iK'$ in this formula and recalled (1.) of § 57:

$$(8.)\qquad \Theta(u+2iK') = -e^{\frac{\pi(K'-iu)}{K}}\Theta(u),$$

this equation results:

$$(9.)\qquad H(u+iK') = ie^{\frac{\pi(K'-2iu)}{4K}}\Theta(u),$$

whence, having again changed u to $u+iK'$, from (7.) it is:

$$(10.)\qquad H(u+2iK') = -e^{\frac{\pi(K'-iu)}{K}}H(u).$$

From the formulas (7.) − (10.) one can derive the more general ones:

$$(11.)\qquad e^{\frac{\pi uu}{4KK'}}\Theta(u) = (-1)^m e^{\frac{\pi(u+2miK')}{4KK'}}\Theta(u+2miK')$$

$$(12.)\qquad e^{\frac{\pi uu}{4KK'}}H(u) = (-1)^m e^{\frac{\pi(u+2miK')}{4KK'}}H(u+2miK')$$

$$(13.)\qquad e^{\frac{\pi uu}{4KK'}}H(u) = (-i)^{2m+1} e^{\frac{\pi(u+(2m+1)iK')}{4KK'}}\Theta(u+(2m+1)iK')$$

$$(14.)\qquad e^{\frac{\pi uu}{4KK'}}\Theta(u) = (-i)^{2m+1} e^{\frac{\pi(u+(2m+1)iK')}{4KK'}}H(u+(2m+1)iK')$$

From (12.), (13.) it is:

$$(15.)\qquad \Theta((2m+1)iK') = 0, \qquad H(2miK') = 0.$$

Formulas (5.), (6.) show that the functions $\Theta(u)$, $H(u)$ having changed u to $u+4K$, formulas (11.), (12.) that the functions

$$e^{\frac{\pi uu}{4KK'}}\Theta(u), \qquad e^{\frac{\pi uu}{4KK'}}H(u),$$

having changed u into $u+4iK'$, remain unchanged; hence these have the real period in common with the elliptic functions, those have the other imaginary period in common with the elliptic functions.

From formula (5.) § 56:

$$\frac{\Theta(iu,k)}{\Theta(0,k)} = e^{\frac{\pi uu}{4KK'}} \cos\operatorname{am}(u,k')\frac{\Theta(u,k')}{\Theta(0,k')}$$

it follows:

$$\frac{H(iu,k)}{\Theta(0,k)} = \sqrt{k}\sin\operatorname{am}(iu,k)\frac{\Theta(iu,k)}{\Theta(0,k)} = ie^{\frac{\pi uu}{4KK'}}\sqrt{k}\sin\operatorname{am}(u,k')\frac{\Theta(u,k')}{\Theta(0,k')},$$

whence from (1.) it is:

$$(16.)\qquad \frac{\Theta(iu,k)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi uu}{4KK'}}\frac{H(u+K',k')}{\Theta(0,k')}$$

$$(17.)\qquad \frac{H(iu,k)}{\Theta(0,k)} = i\sqrt{\frac{k}{k'}}e^{\frac{\pi uu}{4KK'}}\frac{H(u,k')}{\Theta(0,k')}.$$

From (16.) having changed u to iu and commuted k and k' it follows:

$$(18.)\qquad \frac{H(iu+K,k)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi uu}{4KK'}}\frac{\Theta(u,k')}{\Theta(0,k')},$$

to which we add (9.) § 57:

$$(19.)\qquad \frac{\Theta(iu+K,k)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}e^{\frac{\pi uu}{4KK'}}\frac{\Theta(u+K',k')}{\Theta(0,k')}.$$

From the formula found above:

$$\Theta(u+v)\Theta(u-v) = \frac{\Theta^2(u)\Theta^2(v)}{\Theta^2(0)}(1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v)$$

it follows:

$$(20.)\qquad \Theta(u+v)\Theta(u-v) = \frac{\Theta^2(u)\Theta^2(v)-H^2(u)H^2(v)}{\Theta^2(0)}.$$

Having multiplied the formula by:

$$k\sin\operatorname{am}(u+v)\sin\operatorname{am}(u-v) = \frac{k\sin^2\operatorname{am}u - k\sin^2\operatorname{am}v}{1-k^2\sin^2\operatorname{am}u\sin^2\operatorname{am}v} = \frac{H^2(u)\Theta^2(v)-\Theta^2(u)H^2(v)}{\Theta^2(u)\Theta^2(v)-H^2(u)H^2(v)},$$

it results:

$$(21.) \qquad H(u+v)H(u-v) = \frac{H^2(u)\Theta^2(v) - \Theta^2(u)H^2(v)}{\Theta^2(0)}.$$

2.9 On the expansion of the Functions H, Θ into series. The third Expansion of the elliptic Functions.

62.

Let us expand the functions:

$$\frac{\Theta\left(\frac{2kx}{\pi}\right)}{\Theta(0)} = \frac{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}$$
$$\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \frac{2\sqrt[4]{q}\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}$$

into series:

$$\frac{\Theta\left(\frac{2kx}{\pi}\right)}{\Theta(0)} = A - 2A'\cos 2x + 2A''\cos 4x - 2A'''\cos 6x + 2A^{\mathrm{IV}}\cos 8x - \cdots$$
$$\frac{H\left(\frac{2kx}{\pi}\right)}{\Theta(0)} = 2\sqrt[4]{q}[B'\sin x - 2B''\sin 3x + 2B'''\sin 5x - 2B^{\mathrm{IV}}\sin 7x + \cdots].$$

We obtain the determination of A, A', A'', $A'''\cdots$; B', B'', B''', $B^{\mathrm{IV}},\cdots$ by means of equations (7.) $-$ (10.) of the preceding § which having put $U = \frac{2Kx}{\pi}$, $q = e^{\frac{-\pi K'}{K}}$ go over into the following:

$$\begin{aligned}
\Theta\left(\frac{2Kx}{\pi}\right) &= -qe^{2ix}\Theta\left(\frac{2Kx}{\pi} + 2iK'\right)\\
H\left(\frac{2Kx}{\pi}\right) &= -qe^{2ix}H\left(\frac{2Kx}{\pi} + 2iK'\right)\\
i\Theta\left(\frac{2Kx}{\pi}\right) &= +\sqrt[4]{q}e^{ix}H\left(\frac{2Kx}{\pi} + iK'\right)\\
iH\left(\frac{2Kx}{\pi}\right) &= +\sqrt[4]{q}e^{ix}\Theta\left(\frac{2Kx}{\pi} + iK'\right).
\end{aligned}$$

For this aim, we exhibit the propounded expansions this way:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = A - A'e^{2ix} + A''e^{4ix} - A'''e^{6ix} + A^{\mathrm{IV}}e^{8ix} - \cdots$$
$$- A'e^{-2ix} + A''e^{-4ix} - A'''e^{-6ix} + A^{\mathrm{IV}}e^{-8ix} - \cdots$$

$$\frac{iH\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \sqrt[4]{q}[B'e^{ix} - B''e^{3ix} + B'''e^{5ix} - B^{\mathrm{IV}}e^{7ix} + \cdots]$$
$$- \sqrt[4]{q}[B'e^{-ix} - B''e^{-3ix} + B'''e^{-5ix} - B^{\mathrm{IV}}e^{-7ix} + \cdots]$$

Having changed x to $x - i\ln q$ e^{mix} goes over into $q^m e^{mix}$, e^{-mix} goes over into $\frac{e^{-mix}}{q^m}$; further, $\Theta\left(\frac{2Kx}{\pi}\right)$, $H\left(\frac{2Kx}{\pi}\right)$ go over into $\Theta\left(\frac{2Kx}{\pi} + 2iK'\right)$, $H\left(\frac{2Kx}{\pi} + 2iK'\right)$. Hence we obtain:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = -qe^{2ix} \cdot \frac{\Theta\left(\frac{2Kx}{\pi} + 2iK'\right)}{\Theta(0)}$$

$$= \frac{A'}{q} - Aqe^{2ix} + A'q^3e^{4ix} - A''q^5e^{6ix} + A'''q^7e^{8ix} - \cdots$$
$$- \frac{A''}{q^3}e^{-2ix} + \frac{A''}{q^5}e^{-4ix} - \frac{A^{\mathrm{IV}}}{q^7}e^{-6ix} + \frac{A^{\mathrm{V}}}{q^9}e^{-8ix} - \cdots$$

$$\frac{iH\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = -qe^{2ix} \cdot \frac{iH\left(\frac{2Kx}{\pi} + 2iK'\right)}{\Theta(0)}$$

$$= \sqrt[4]{q}\left\{B'e^{ix} - B'q^2e^{3ix} + B''q^4e^{5ix} - B'''q^6e^{7ix} + \cdots\right\}$$
$$- \sqrt[4]{q}\left\{\frac{B''}{q^2}e^{-ix} - \frac{B'''}{q^4}e^{-3ix} + \frac{B^{\mathrm{IV}}}{q^6}e^{-5ix} - \frac{B^{\mathrm{V}}}{q^8}e^{-7ix} + \cdots\right\}.$$

Having compared those to the propounded expressions we find:

$$\begin{array}{llll} A' = Aq, & A'' = A'q^3, & A''' = A''q^5, & A^{\mathrm{IV}} = A'''q^7, \cdots, \\ B'' = B'q^2, & B''' = B''q^4, & B^{\mathrm{IV}} = B'''q^6, & B^{\mathrm{V}} = B^{\mathrm{IV}}q^8, \cdots, \end{array}$$

and hence:

$$\begin{array}{llll} A' = Aq, & A'' = Aq^4, & A''' Aq^9, & A^{\mathrm{IV}} = Aq^{16}, \cdots, \\ B'' = B'q^2, & B''' = B'q^6, & B^{\mathrm{IV}} B'q^{12}, & B^{\mathrm{V}} = B'q^{20}, \cdots, \end{array}$$

whence the expansions in question become:

$$\begin{aligned} \frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} &= A[1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots] \\ \frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} &= 2\sqrt[4]{q}B'[\sin x - q^2\sin 3x + q^{2\cdot 3}\sin 5x - q^{3\cdot 4}\sin 7x + q^{4\cdot 5}\sin 9x - \cdots] \\ &= B'[2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots]. \end{aligned}$$

The found expansions could have been derived from each other by means of the formula:

$$iH\left(\frac{2Kx}{\pi}\right) = \sqrt[4]{q}e^{ix}\Theta\left(\frac{2Kx}{\pi} + iK'\right).$$

For, having found the series:

$$\frac{\Theta\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = A[1 - q(e^{2ix} + e^{-2ix}) + q^4(e^{4ix} + e^{-4ix}) - q^9(e^{6ix} + e^{-6ix}) + \cdots],$$

by changing x to $x - i\ln\sqrt{q}$ having done which e^{2mix}, e^{-2imx} go over into $q^m e^{2imx}$, $\frac{e^{-2mix}}{q^m}$, $\Theta\left(\frac{2Kx}{\pi}\right)$ into $\Theta\left(\frac{2Kx}{\pi} + iK'\right)$, and by multiplying by $\sqrt[4]{q}e^{ix}$ we obtain:

$$\begin{gathered} \frac{iH\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = \sqrt[4]{q}e^{ix}\frac{\Theta\left(\frac{2Kx}{\pi} + iK'\right)}{\Theta(0)} \\ = A[\sqrt[4]{q}(e^{ix} - e^{-ix}) - \sqrt[4]{q^9}(e^{3ix} - e^{-3ix}) + \sqrt[4]{q^{25}}(e^{5ix} - e^{-5ix}) - \cdots] \end{gathered}$$

or:

$$\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta(0)} = A[2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots].$$

Additionally, by this analysis we find:

$$B' = A.$$

63.

The determination of A demands particular artifices. Let, as it is possible from the preceding, us put:

$$\begin{aligned}
&(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots \\
&= P(q)[1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots] \\
&\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots \\
&= P(q)[\sin x-q^{1\cdot 2}\sin 3x+q^{2\cdot 3}\sin 5x-q^{3\cdot 4}\sin 7x+q^{4\cdot 5}\sin 9x-\cdots];
\end{aligned}$$

it is:

$$A = \frac{P(q)}{[(1-q)(1-q^3)(1-q^5)\cdots]^2}.$$

The second expression remains unchanged, if it is multiplied by the first and after that it is put q^2 instead of q. Hence we obtain the identity:

$$\begin{aligned}
&P(q^2)P(q^2)[\sin x-q^4\sin 3x+q^{12}\sin 5x-q^{24}\sin 7x+\cdots] \\
&\times[1-2q^2\cos 2x+2q^8\cos 4x-2q^{18}\cos 6x+\cdots] \\
&= P(q)[\sin x-q^2\sin 3x+q^6\sin 5x-q^{12}\sin 7x+\cdots].
\end{aligned}$$

Now, let us do the multiplication explicitly, such that one writes $\sin(m+n)x + \sin(m-n)x$ instead of $2\sin mx\cos nx$ everywhere: It is easily seen that the coefficients of $\sin x$ in the expanded product will be:

$$1+q^2+q^6+q^{12}+q^{20}+\cdots,$$

such that this equation results:

$$\frac{P(q)}{P(q^2)P(q^2)} = 1+q^2+q^6+q^{12}+q^{20}+\cdots$$

But from the second of the propounded formulas having put $x=\frac{\pi}{2}$ we find:

$$[(1+q^2)(1+q^4)(1+q^6)\cdots]^2 = P(q)[1+q^2+q^6+q^{12}+q^{20}+\cdots],$$

whence:

$$\frac{P(q)P(q)}{P(q^2)P(q^2)} = [(1+q^2)(1+q^4)(1+q^6)\cdots]^2$$

or:

$$\begin{aligned}\frac{P(q)}{P(q^2)} &= (1+q^2)(1+q^4)(1+q^6)\cdots \\ &= \frac{(1-q^4)(1-q^8)(1-q^{12})\cdots}{(1-q^2)(1-q^4)(1-q^6)\cdots}.\end{aligned}$$

Hence it finally results:

$$\begin{aligned}A &= \frac{1}{(1-q^2)(1-q^4)(1-q^6)\cdots}\cdot\frac{1}{[(1-q)(1-q^3)(1-q^5)\cdots]^2} \\ &= \frac{(1+q)(1+q^2)(1+q^3)(1+q^4)\cdots}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}\end{aligned}$$

or from those expansions we found in § 36:

$$\frac{1}{A} = \sqrt{\frac{2k'K}{\pi}}.$$

That quantity we left undetermined up to this point we now want to put $\Theta(0)$:

$$\Theta(0) = \frac{1}{A} = \sqrt{\frac{2k'K}{\pi}},$$

it is found:

(1.) $$\Theta\left(\frac{2Kx}{\pi}\right) = 1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots,$$

(2.) $$H\left(\frac{2Kx}{\pi}\right) = 2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots$$

64.

It is convenient to investigate the identity we proved in the last paragraph:

$$
\begin{aligned}
&(1-2q\cos 2x+q^2)(1-2q^3\cos 3x+q^6)(1-2q^5\cos 2x+q^{10})\cdots \\
&=\frac{1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}
\end{aligned}
$$

in another way completely different from the preceding. For this aim, we want to mention the following two formulas as lemmas:

$$
(1.)\qquad (1+qz)(1+q^3z)(1+q^5z)(1+q^7)\cdots
$$

$$
=1+\frac{qz}{1-q^2}+\frac{q^4z^2}{(1-q^2)(1-q^4)}+\frac{q^9z^3}{(1-q^2)(1-q^4)(1-q^6)}+\frac{q^{16}z^4}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)}+\cdots
$$

$$
(2.)\qquad \frac{1}{(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)\cdots}
$$

$$
\begin{aligned}
&=1+\frac{q}{1-q}\cdot\frac{z}{1-qz}+\frac{q^4}{(1-q)(1-q^2)}\cdot\frac{z^2}{(1-qz)(1-q^2z)} \\
&+\frac{q^2}{(1-q)(1-q^2)(1-q^3)}\cdot\frac{z^3}{(1-qz)(1-q^2z)(1-q^3z)}+\cdots
\end{aligned}
$$

For the demonstration of the first I observe that the expression:

$$
(1+qz)(1+q^3z)(1+q^5z)(1+q^7z)\cdots,
$$

having put q^2z instead of z and having multiplied by $(1+qz)$ remains unchanged; hence having put:

$$
(1+qz)(1+q^3z)(1+q^5z)\cdots=1+A'z+A''z^2+A'''z^3+\cdots,
$$

it is found:

$$
1+A'z+A''z^2+A'''z^3+\cdots=(1+qz)(1+A'q^2z+A''q^4z^2+A'''z^3+\cdots)
$$

and hence having done the expansion:

$$
A'=q+q^2A',\quad A''=q^3A'+q^4A'',\quad A'''=q^5A''+q^6A''',\cdots
$$

or:

$$A' = \frac{q}{1-q}, \quad A'' = \frac{q^3 A'}{1-q^4}, \quad A''' = \frac{q^5 A''}{1-q^6}, \cdots,$$

whence it is:

$$A' = \frac{q}{1-q}, \quad A'' = \frac{q^4}{(1-q^2)(1-q^4)}, \quad A''' = \frac{q^9}{(1-q^2)(1-q^4)(1-q^6)}, \cdots,$$

as it is propounded.

For the demonstration of formula (2.) I observe that the expression:

$$\frac{1}{(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)\cdots},$$

having put qz instead of z and multiplied by $\frac{1}{1-qz}$ remains unchanged; whence having put:

$$\frac{1}{(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)\cdots}$$
$$= 1 + \frac{A'z}{1-qz} + \frac{A''z^2}{(1-qz)(1-q^2z)} + \frac{A'''z^3}{(1-qz)(1-q^2z)(1-q^3z)} + \cdots,$$

we obtain:

$$1 + \frac{A'z}{1-qz} + \frac{A''z^2}{(1-qz)(1-q^2z)} + \frac{A'''z^3}{(1-qz)(1-q^2z)(1-q^3z)} + \cdots$$
$$= \frac{1}{1-qz} + \frac{A'qz}{(1-qz)(1-q^2z))} + \frac{A''q^2z^2}{(1-qz)(1-q^2z)(1-q^3z)} + \frac{A'''q^3z^3}{(1-qz)(1-q^2z)(1-q^3z)(1-q^4z)} + \cdots$$
$$= 1 + \frac{(q+A'q)z}{1-qz} + \frac{(q^3A' + q^2A'')z^2}{(1-qz)(1-q^2z)} + \frac{(q^5A'' + q^3A''')z^3}{(1-qz)(1-q^2z)(1-q^3z)} + \cdots$$

Hence it follows:

$$A' = q + A'q, \quad A'' = q^3A' + q^2A'', \quad A''' = q^5A'' + q^3A''', \cdots$$

and hence:

$$A' = \frac{q}{1-q}, \quad A'' = \frac{q^3A'}{1-q^2}, \quad A''' = \frac{q^5A''}{1-q^3}, \cdots$$

whence:

$$A' = \frac{q}{1-q}, \quad A'' = \frac{q^4}{(1-q)(1-q^2)}, \quad A''' = \frac{q^9}{(1-q)(1-q^2)(1-q^3)}, \cdots,$$

as it was propounded.

Now, let us form the product:

$$\left\{(1+qz)(1+q^3z)(1+q^5z)\cdots\right\}\left\{\left(1+\frac{q}{z}\right)\left(1+\frac{q^3}{z}\right)\left(1+\frac{q^5}{z}\right)\right\}$$

$$= \left\{1 + \frac{q}{1-q}z + \frac{q^4}{(1-q^2)(1-q^4)}z^2 + \frac{q^9}{(1-q^2)(1-q^4)(1-q^6)}z^3 + \cdots\right\}$$

$$\times \left\{1 + \frac{q}{1-q}\frac{1}{z} + \frac{q^4}{(1-q^2)(1-q^4)}\frac{1}{z^2} + \frac{q^9}{(1-q^2)(1-q^4)(1-q^6)}\frac{1}{z^3} + \cdots\right\}.$$

We find the following value for the coefficient of z^n or even $\frac{1}{z^n}$ which we want to put $B^{(n)}$:

$$B^{(n)} = \frac{q^{nn}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}$$

$$\times \left\{\begin{array}{c} 1 + \frac{q^2}{1-q^2}\cdot\frac{q^{2n}}{1-q^{2n+2}} + \frac{q^8}{(1-q^2)(1-q^4)}\cdot\frac{q^{4n}}{(1-q^{2n+2})(1-q^{2n+4})} \\ + \frac{q^{18}}{(1-q^2)(1-q^4)(1-q^6)}\cdot\frac{q^{6n}}{(1-q^{2n+2})(1-q^{2n+4})(1-q^{2n+6})} + \cdots \end{array}\right\}$$

But from formula (2.) having put q^2 instead of q and $z = q^{2n}$ which is seen in the braces we find:

$$= \frac{1}{(1-q^{2n+2})(1-q^{2n+4})(1-q^{2n+6})(1-q^{2n+8})\cdots},$$

whence it is:

$$B^{(n)} = \frac{q^{nn}}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}$$

and hence:

$$\left\{(1+qz)(1+q^3z)(1+q^5q)\cdots\right\}\left\{\left(1+\frac{q}{z}\right)\left(1+\frac{q^3}{z}\right)\left(1+\frac{q^5}{z}\right)\right\}$$

$$= \frac{1 + q\left(z + \frac{q}{z}\right) + q^4\left(z^2 + \frac{q}{z^2}\right) + q^9\left(z +^3 \frac{q}{z^3}\right) + \cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}$$

or having put $z = e^{2ix}$ and changed q to $-q$ it is:

$$(1 - 2q\cos 2x + q^2)(1 - 2q^3\cos 2x + q^6)(1 - 2q^5\cos 2x + q^{10})\cdots$$
$$= \frac{1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + \cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}.$$

What was to be demonstrated.

If one puts $-qz^2$ instead of z and multiplies by $\sqrt[4]{q}z$, this equation results:

$$\sqrt[4]{q}\left(z - \frac{1}{z}\right)\left\{(1-q^2z^2)(1-q^4z^2)(1-q^6z^2)\cdots\right\}\left\{\left(1-\frac{q^2}{z^2}\right)\left(1-\frac{q^4}{z^2}\right)\left(1-\frac{q^6}{z^2}\right)\cdots\right\}$$
$$= \frac{\sqrt[4]{q}\left(z - \frac{1}{z}\right) - \sqrt[4]{q^9}\left(z^3 - \frac{1}{z^3}\right) + \sqrt[4]{q^{25}}\left(z^5 - \frac{1}{z^5}\right) - \cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}$$

or having put $z = e^{ix}$:

$$2\sqrt[4]{q}\sin x(1 - 2q^2\cos 2x + q^4)(1 - 2q^4\cos 2x + q^8)(1 - 2q^6\cos 2x + q^{12})\cdots$$
$$= \frac{2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}$$

which is the other expansion we found.

65.

The expansions of the functions:

(1.) $$\Theta\left(\frac{2Kx}{\pi}\right) = 1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots$$

(2.) $$H\left(\frac{2Kx}{\pi}\right) = 2\sqrt[4]{q}\sin x - \sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - \sqrt[4]{q^{49}}\sin 7x + \cdots$$

immediately lead to a new expansion of elliptic functions. For, from formulas (1.) by putting $u = \frac{2Kx}{\pi}$ we obtain:

$$\begin{aligned}
\sin\operatorname{am}\frac{2Kx}{\pi} &= \frac{1}{\sqrt{k}}\cdot\frac{H\left(\frac{2Kx}{\pi}\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}\\
\cos\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{\frac{k'}{k}}\cdot\frac{H\left(\frac{2K}{\pi}\left(x+\frac{\pi}{2}\right)\right)}{\Theta\left(\frac{2Kx}{\pi}\right)}\\
\Delta\operatorname{am}\frac{2Kx}{\pi} &= \sqrt{k'}\cdot\frac{\Theta\left(\frac{2K}{\pi}\left(x+\frac{\pi}{2}\right)\right)}{\Theta\left(\frac{2Kx}{\pi}\right)},
\end{aligned}$$

whence:

$$\begin{aligned}
&(3.)\quad \sin\frac{2Kx}{\pi} = \frac{1}{\sqrt{k}}\cdot\frac{2\sqrt[4]{q}\sin x - 2\sqrt[4]{q^9}\sin 3x + 2\sqrt[4]{q^{25}}\sin 5x - 2\sqrt[4]{q^{49}}\sin 7x + \cdots}{1-2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots}\\
&(4.)\quad \cos\frac{2Kx}{\pi} = \sqrt{\frac{k'}{k}}\cdot\frac{2\sqrt[4]{q}\cos x + 2\sqrt[4]{q^9}\cos 3x + 2\sqrt[4]{q^{25}}\cos 5x + 2\sqrt[4]{q^{49}}\cos 7x + \cdots}{1-2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots}\\
&(5.)\quad \Delta\operatorname{am}\frac{2Kx}{\pi} = \sqrt{k'}\cdot\frac{1+2q\cos 2x + 2q^4\cos 4x + 2q^9\cos 6x + 2q^{16}\cos 8x + \cdots}{1-2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + 2q^{16}\cos 8x - \cdots}.
\end{aligned}$$

Further, from (2.), (3.) of § 61, because it was put $\Theta(0) = \sqrt{\frac{2k'K}{\pi}}$, we obtain:

$$\Theta(K) = \sqrt{\frac{2K}{\pi}},\quad H(K) = \sqrt{\frac{2kK}{\pi}},\quad \Theta(0) = \sqrt{\frac{2k'K}{\pi}},\quad H'(0) = \sqrt{\frac{2kk'K}{\pi}},$$

whence from (1.), (2.) it follows:

$$\begin{aligned}
&(6.)\quad \sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + \cdots\\
&(7.)\quad \sqrt{\frac{2kK}{\pi}} = 2\sqrt[4]{q} + 2\sqrt[4]{q^9} + 2\sqrt[4]{q^{25}} + 2\sqrt[4]{q^{49}} + 2\sqrt[4]{q^{81}} + \cdots\\
&(8.)\quad \sqrt{\frac{2k'K'}{\pi}} = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - 2q^{25} + \cdots\\
&(9.)\quad \sqrt{kk'\left(\frac{2K}{\pi}\right)^3} = 2\sqrt[4]{q} - 6\sqrt[4]{q^9} + 10\sqrt[4]{q^{25}} - 14\sqrt[4]{q^{49}} + 18\sqrt[4]{q^{81}} - \cdots,
\end{aligned}$$

whence it also is:

$$(10.)\quad \sqrt{k} = \frac{2\sqrt[4]{q}+1\sqrt[4]{q^9}+2\sqrt[4]{q^{25}}+2\sqrt[4]{q^{49}}+2\sqrt[4]{q^{81}}+\cdots}{1+2q+2q^4+2q^9+2q^{16}+\cdots}$$

$$(11.)\quad \sqrt{k'} = \frac{1-2q+2q^4-2q^9+2q^{16}-2q^{25}+\cdots}{1+2q+2q^4+2q^9+2q^{16}+2q^{25}+\cdots}.$$

Further, because $Z(u)=\frac{\Theta'(u)}{\Theta(u)}$ and $\Pi(u,a)=uZ(a)+\frac{1}{2}\ln\frac{\Theta(u-a)}{\Theta(u+a)}$, it is:

$$(12.)\quad \frac{2K}{\pi}\cdot Z\left(\frac{2Kx}{\pi}\right) = \frac{4q\sin 2x-8q^4\sin 4x+12q^9\sin 6x-16q^{16}\sin 8x+\cdots}{1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots}$$

$$(13.)\quad \Pi\left(\frac{2Kx}{\pi},\frac{2KA}{\pi}\right)$$

$$=\frac{2Kx}{\pi}\cdot Z\left(\frac{2KA}{\pi}\right)+\frac{1}{2}\ln\frac{1-2q\cos 2(x-A)+2q^4\cos 4(x-A)-2q^9\cos 6(x-A)+\cdots}{1-2q\cos 2(x+A)+2q^4\cos 4(x+A)-2q^9\cos 6(x+A)+\cdots}.$$

This is the third expansion of elliptic functions.

66.

From the found expansions:

$$(1.)\quad [(1-q^2)(1-q^4)(1-q^6)\cdots](1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots$$
$$=1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+2q^{16}\cos 8x-\cdots$$
$$[(1-q^2)(1-q^4)(1-q^6)\cdots]\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots$$
$$=\sin x-q^2\sin 3x+q^6\sin 5x-q^{12}\sin 7x+q^{20}\sin 9x-\cdots,$$

the second of which having put $\sqrt{q}$ instead of q can also be exhibited as this:

$$(2.)\quad [(1-q)(1-q^2)(1-q^3)\cdots]\sin(1-2q\cos 2x+q^2)(1-2q^2\cos 2x+q^4)(1-2q^3\cos 2x+q^6)\cdots$$
$$=\sin x-q\sin 3x+q^3\sin 5x-q^6\sin 7x+q^{10}\sin 9x-q^{15}\sin 11x+\cdots,$$

having put $x=0$, $x=\frac{\pi}{2}$ it follows:

$$(3.)\quad \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}{(1+q)(1+q^2)(1+q^3)(1+q^4)\cdots} = 1-2q+2q^4-2q^9+2q^{16}-\cdots$$

$$(4.)\quad \frac{(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots}{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots} = 1+q+q^3+q^6+q^{10}+q^{15}+\cdots$$

$$(5.)\quad [(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots]^3 = 1-3q+5q^3-7q^6+9q^{10}-\cdots$$

Let us in (2.) put $x = \frac{\pi}{3}$, it is $\sin x = +\sqrt{\frac{3}{4}}$, $\sin 3x = 0$, $\sin 5x = -\sqrt{\frac{3}{4}}$, $\sin 7x = +\sqrt{\frac{3}{4}}$, etc.; further, $(1-q)(1-2q\cos 3x + q^2) = 1 - q^3$, whence (2.) goes over into this formula:

$$(1-q^3)(1-q^6)(1-q^9)(1-q^{12})\cdots = 1 - q^3 - q^6 + q^{15} + q^{21} - q^{36} - \cdots$$

or:

$$(6.)\qquad (1-q)(1-q^2)(1-q^3)(1-q^4)\cdots = 1 - q - q^2 + q^5 + q^7 - q^{12} - \cdots,$$

the general term of which series is:

$$(-1)^n q^{\frac{3nn \pm n}{2}}.$$

Having compared (5.) and (6.) to each other we obtain:

$$(7.)\qquad [1 - q - q^2 + q^5 + q^7 - q^{12} - \cdots]^3 = 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \cdots.$$

Formula (4.) was also found by Gauss in the paper: *Summatio serierum quarundam singularium.* Comm. Gott. Vol. I. a. 1808-1811. He deduced it from the following memorable formula:

$$(8.)\qquad \frac{(1-qz)(1-q^3z)(1-q^5z)(1-q^7z)\cdots}{(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots}$$
$$= 1 + \frac{q(1-z)}{1-q} + \frac{q^3(1-z)(1-qz)}{(1-q)(1-q^2)} + \frac{q^6(1-z)(1-qz)(1-q^2z)}{(1-q)(1-q^2)(1-q^3)} + \cdots,$$

having put $z = q$. To these one can add other similar formulas whose proof I omit here:

$$(9.)\qquad \frac{1}{2}\frac{(1+z)(1+qz)(1+q^2z)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots} + \frac{1}{2}\frac{(1-z)(1-qz)(1-q^2z)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots}$$
$$= 1 - \frac{q(1-z^2)}{1-q^2} + \frac{q^4(1-z^2)(1-q^2z^2)}{(1-q^2)(1-q^4)} - \frac{q^9(1-z^2)(1-q^2z^2)(1-q^4z^2)\cdots}{(1-q^2)(1-q^4)(1-q^6)} + \cdots$$

$$(10.)\qquad \frac{q}{2z}\frac{(1+z)(1+qz)(1+q^2z)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots} - \frac{q}{2z}\frac{(1-z)(1-qz)(1-q^2z)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots}$$

$$= q - \frac{q^4(1-z^2)}{1-q^2} + \frac{q^9(1-z^2)(1-q^2z^2)}{(1-q^2)(1-q^4)} - \frac{q^{16}(1-z^2)(1-q^2z^2)(1-q^4z^2)\cdots}{(1-q^2)(1-q^4)(1-q^6)} + \cdots$$

of which (9.) having put $z = q$ yields:

$$\frac{1}{2} + \frac{1}{2}\frac{(1-q)(1-q^2)(1-q^3)\cdots}{(1+q)(1+q^2)(1+q^3)\cdots} = 1 - q + q^4 - q^9 + \cdots$$

or:

$$\frac{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}{(1+q)(1+q^2)(1+q^3)(1+q^4)\cdots} = 1 - 2q + 2q^4 - 2q^9 + \cdots,$$

which is formula (3.).

Formula (6.) which is very deep and the one depending on the trisection of elliptic functions was already found by Euler a long time ago and proved lucidly. This extraordinary proof is to be treated on another occasion in more detail.

To these we add the following expansions:

$$(11.)\qquad \frac{\sqrt{kk'\left(\frac{2K}{\pi}\right)^3}}{\Theta\left(\frac{2kx}{\pi}\right)} = \frac{2\sqrt[4]{q}[(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots]^2}{(1-2q\cos 2x+q^2)(1-2q^3\cos 2x+q^6)(1-2q^5\cos 2x+q^{10})\cdots}$$

$$= \frac{2\sqrt[4]{q}(1-q^2)}{1-2q\cos 2x+q^2} - \frac{2\sqrt[4]{q^9}(1-q^6)}{1-2q^3\cos 2x+q^6} + \frac{2\sqrt[4]{q^{25}}(1-q^{10})}{1-2q^5\cos 2x+q^{10}} - \cdots$$

$$(12.)\qquad \frac{\sqrt{kk'\left(\frac{2K}{\pi}\right)^3}}{H\left(\frac{2kx}{\pi}\right)} = \frac{[(1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots]^2}{\sin x(1-2q^2\cos 2x+q^4)(1-2q^4\cos 2x+q^8)(1-2q^6\cos 2x+q^{12})\cdots}$$

$$= \frac{1}{\sin x} - \frac{4q^2(1+q^2)\sin x}{1-2q^2\cos 2x+q^4} + \frac{4q^6(1+q^4)\sin x}{1-2q^4\cos 2x+q^8} - \frac{4q^{12}(1+q^6)\sin x}{1-2q^6\cos 2x+q^{12}} + \cdots$$

$$= \frac{1}{\sin x}\left\{\frac{(1-q^2)(1-q^4)}{1-2q^2\cos 2x+q^4} - \frac{q^2(1-q^4)(1-q^8)}{1-2q^4\cos 2x+q^8} + \frac{q^6(1-q^6)(1-q^{12})}{1-2q^6\cos 2x+q^{12}} - \cdots\right\},$$

which are easily obtained from the known theory of composite fractions into simple ones.

Hence one deduces the special expansions:

$$(13.)\qquad \frac{2kK}{\pi} = 4\sqrt{q}\left(\frac{1+q^2}{1-q^2}\right) - 4\sqrt{q^9}\left(\frac{1+q^6}{1-q^6}\right) + 4\sqrt{q^{25}}\left(\frac{1+q^{10}}{1-q^{10}}\right) - \cdots$$

$$(14.)\qquad \frac{2k'K}{\pi} = 1 - \frac{4q}{1+q} + \frac{4q^3}{1+q^2} - \frac{4q^6}{1+q^3} + \frac{q4^{10}}{1+q^4} - \cdots$$

Having compared these expansions of the expressions $\frac{2kK}{\pi}$, $\frac{2k'K}{\pi}$ exhibited above these equations result:

$$\frac{\sqrt{q}}{1-q} - \frac{\sqrt{q^3}}{1-q^3} + \frac{\sqrt{q^5}}{1-q^5} - \frac{\sqrt{q^7}}{1-q^7} + \cdots = \sqrt{q}\left(\frac{1+q^2}{1-q^2}\right) - \sqrt{q^9}\left(\frac{1+q^6}{1-q^6}\right) + \sqrt{q}\left(\frac{1+q^{10}}{1-q^{10}}\right) - \cdots$$

$$1 - \frac{4q}{1+q} + \frac{4q^3}{1+q^3} - \frac{4q^5}{1+q^5} + \frac{4q^7}{1+q^7} - \cdots = 1 - \frac{4q}{1+q} + \frac{4q^3}{1+q^2} - \frac{4q^6}{1+q^3} + \frac{4q^{10}}{1+q^4} - \cdots$$

In like manner, Clausen recently observed that the series:

$$\frac{q}{1-q} + \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} + \frac{q^4}{1-q^4} + \cdots$$

can be transformed into:

$$q\left(\frac{1+q}{1-q}\right) + q^4\left(\frac{1+q^2}{1-q^2}\right) + q^9\left(\frac{1+q^3}{1-q^3}\right) + q^{16}\left(\frac{1+q^4}{1-q^4}\right) + \cdots$$

Above we found the expansions of $\frac{2K}{\pi}$, $\frac{2kK}{\pi}$ and their second, third, fourth powers into series. Therefore, these yield expansions of the second, fourths, sixth and eighth powers of the expressions:

$$\sqrt{\frac{2K}{\pi}} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots$$

$$\sqrt{\frac{2kK}{\pi}} = 2\sqrt[4]{q} + 2\sqrt[4]{q^9} + 2\sqrt[4]{q^{25}} + 2\sqrt[4]{q^{49}} + \cdots$$

whence various arithmetical theorems follow. So, for the sake of an example, from the formula:

$$
\begin{aligned}
\left(\frac{2K}{\pi}\right)^2 &= \left\{1+2q+2q^4+2q^9+2q^{16}+\cdots\right\}^4 \\
&= 1+8\left\{\frac{q}{1-q}+\frac{2q^2}{1+q^2}+\frac{3q^3}{1-q^3}+\frac{4q^4}{1+q^4}+\cdots\right\} \\
&= 1+8\sum\varphi(p)\left\{q^p+3q^{2p}+3q^{4p}+3q^{8p}+\cdots\right\},
\end{aligned}
$$

where $\varphi(p)$ is an arbitrary odd number, $\varphi(p)$ is the sum of factors of p, the famous Fermat-Theorem follows as a corollary, namely, that any number is the sum of four squares.